DEPARTMENT OF EDUCATION
PROVINCE OF ALBERTA

LOCUS

AND

The CIRCLE

Section B: Mathematics 20

By

HENRY BOWERS
NORMAN MILLER
ROBERT E. K. ROURKE

CURRICULUM
QA
484
B78

DENT : MACMILLAN

CURR HIST

Copyright, Canada, 1950

by

J. M. DENT AND SONS (CANADA) LTD.

and

THE MACMILLAN COMPANY OF CANADA LIMITED

All rights reserved—no part of this book may be reproduced in any form without the written permission of the publisher.

Mimeographing or reproducing mechanically in any other way passages from this book without the written permission of the publisher is an infringement of the copyright law.

PRINTED IN CANADA

LOCUS AND THE CIRCLE

LOCUS

Everyone has watched a honey-bee as it flits from flower to flower in the garden. Imagine, if you can, what a wire model of its path would look like. * * * No doubt you have watched a rural mail-delivery man making his rounds in the country. Think of a diagram of his path as he covers his daily route. * * * What would be the path of a speck on a window-blind which is being raised?

Of all the examples of paths which you can bring to mind, some can be easily and precisely described. You will have no difficulty in giving the description of each of the following paths:

(1) The path of a pin point on a drawer which is being pulled out of a desk.

(2) The path of a speck at the end of a hand of a watch lying on a table.

(3) The path of a speck on the head of a nail on a barn door as the door swings on its hinges.

(4) The path of a point on the seat of a swing when a child swings on it.

(5) The path of the centre of a wheel which is rolling along a straight line.

(6) The path of the centre of a wheel which is rolling around a level circular track.

Here is a harder one for you to exercise your wits on:

The path in space of a fly which walks in a straight line across a window-blind which is being raised at a uniform rate.

All these paths are examples of ***loci.*** (*Locus*, the singular, is pronounced to rhyme with 'woke us'; *loci*, the plural, is pronounced 'low sigh'.)

A path traced by a moving point is a locus.

In each of the examples just given, it is possible to state a law which the moving point obeys. Thus, in (1), assuming that the bottom of the drawer remains horizontal, the point must move so that it is always at a fixed distance from the floor.

LIBRARY OF THE UNIV
OF ALBERTA

Hence, if the drawer does not 'wobble' either to the left or to the right, the path of the point is a straight line parallel to the floor.

In (2), the law is that the speck, during its motion, must a main in the same plane and must keep at a fixed distance from refixed point. Hence, the path of the speck is a circle.

Let us consider another example: if a performing flea is attached by a leash to a ring on a horizontal wire above a table, what is the locus of its path on the table when it moves so as to keep the leash taut?

The law has been stated. The path is a closed figure with two straight sides and two semicircular ends.

We shall now exclude from our study irregular paths like that of the honey-bee, and restrict ourselves to those for which a law can be stated.

> *The locus of a point is the path traced by the point when it moves in accordance with a given law.*

EXERCISES

(Proofs are not required)

1. What is the locus of a point which moves in a plane so that it is always 1 inch away from a fixed point in the same plane?

2. (*a*) What is the locus of a point which moves in a plane so that it is always 5 centimetres from a given straight line and on one side of the line?

(*b*) What difference would be made in the answer to (*a*) if the words 'on one side of the line' were omitted?

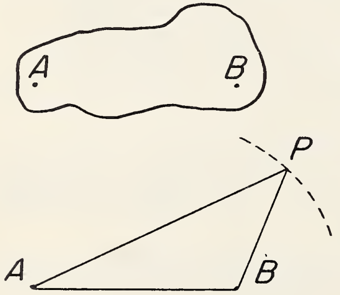
N.B. In the remainder of this chapter, unless there is a statement to the contrary, it will be understood that the locus is confined to a plane.

3. Give the locus of a point which moves so that it is equidistant from two fixed parallel lines.

4. A twenty-five cent piece is rolled around the circumference of a stationary fifty-cent piece, and always in contact with it. What is the locus of the centre of the twenty-five cent piece?

5. A and B are two fixed points. P is a point which moves so that its distance from A is always the same as its distance from B. Using instruments, find a number of positions of P. What line do you suspect to be the locus of P?

6. A and B are two fixed points. P is a point which moves so that $PA + PB$ is a fixed length. What is the locus of P? (*Hints of immense strength:* On your paper, take two points A and B, say 3 inches apart. Draw a line 4 inches long to represent the constant distance $PA + PB$. Use your instruments to find a number of positions of P. One position will be on AB produced; another will be on BA produced. There will be positions on both sides of AB. On what kind of line does P appear to travel? * * * The curve is called an ***ellipse***.)



After doing this exercise, you may be interested in drawing an ellipse in the following manner. Insert a thumb tack at A and another at B. Take a closed loop of thread and, with a pencil as indicated in the adjoining figure, trace a line. The curve formed is an ellipse. Note that $PA + PB$ is a constant length.

* * *

You have now had some experience in the practical method of finding a locus. With the aid of your instruments, you obtained

various positions of the moving point and arrived at a conclusion as to what the locus is *likely* to be.

How to Give a Theoretical Proof that a Supposed Locus is the True Locus

A locus guessed at, or inferred, from a set of positions determined with the aid of instruments must be regarded as conjectural until a proof is given.

A theoretical proof must establish two *conclusions:*

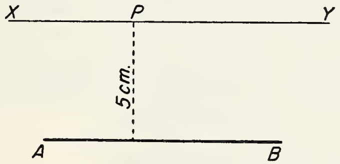
1. It must show that, ***if a point obeys the given law, it lies on the line (or lines) forming the supposed locus.***

2. It must show that, ***if a point lies on the line (or lines) forming the supposed locus, it obeys the given law.***

You will notice that *each of these two statements is the converse of the other.*

Let us now consider Exercise 2 (a) on page 377.

What is the locus of a point which moves so that it is always 5 centimetres from a given straight line and on one side of the line?



After drawing the figure, no doubt you concluded that the, ocus of the point (P) is the straight line XY parallel to, and 5 centimetres distant from, the given line AB. In other words XY is the supposed or conjectural locus.

To prove that XY is really the locus of P, it would be necessary to show that both of the following statements are true:

1. If any point obeys the law (that it is always on one side of, and 5 centimetres distant from, AB), then it must lie on XY.

2. If any point lies on XY, then it must obey the law (namely, that it is always on one side of, and 5 centimetres distant from, AB).

Unless both statements are proved, we cannot say that (1) *all points obeying the law lie on* XY, *and that* (2) *all points lying on* XY *obey the law.*

If we omitted the proof of (1), it might be possible that some points obeying the law did not lie on XY.

If we omitted the proof of (2), it might be possible that one or more points on XY did not obey the law.

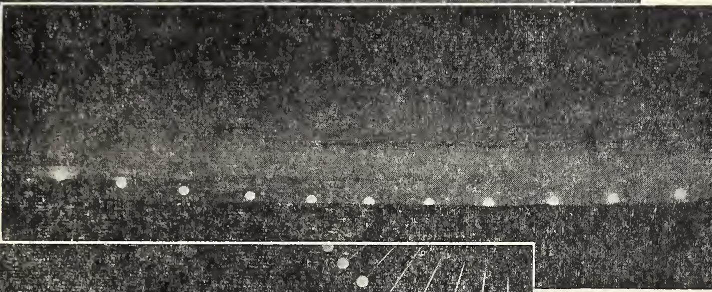
Thus, we can give now a more complete description of a locus:

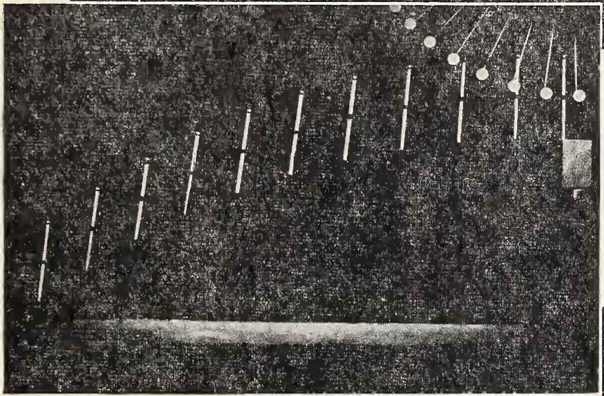
The locus of a point is the path traced by the point when it moves in accordance with a given law.

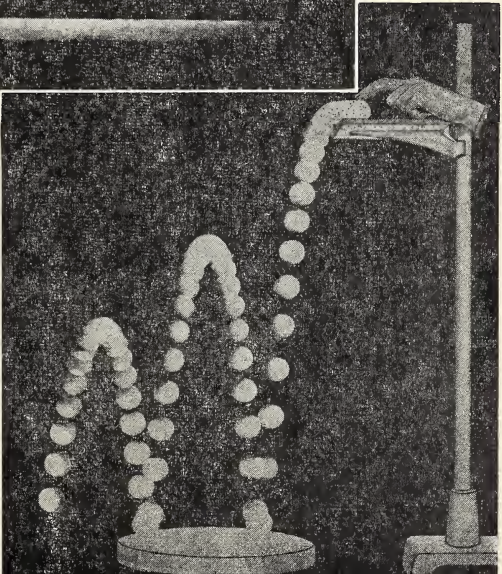
All points satisfying the law lie on the locus.

All points lying on the locus satisfy the law.

As an example, let us prove that a supposed locus is actually the locus.



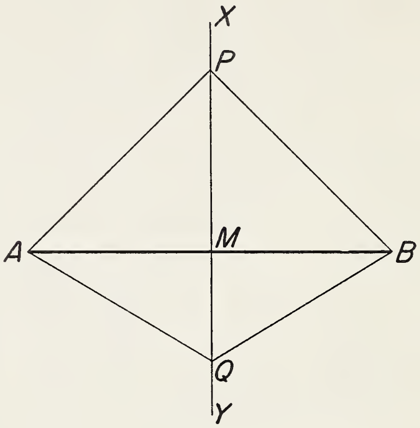




These multiple-exposure photographs illustrate some interesting loci: (i) and (ii) path of the midnight sun; (iii) simple pendulum striking stick. (*Courtesy Prof. F. W. Sears and H. E. Egerton, Mass. Inst. of Technology*); (iv) bouncing golf ball. (*From Sears' Principles of Physics, Vol. I, by permission of Addison-Wesley Press, Inc., publishers.*)

PROPOSITION 4. THEOREM

If a point moves so that it is equidistant from two fixed points, its locus is the right bisector of the straight line joining the two points.



Given: A and B are two fixed points. P is any point equidistant from A and B.

Required: To prove that the locus of P is the right bisector of AB.

Proof: Bisect AB at M.
Join PA, PB, PM and produce PM to X and Y.

(1) *To prove that any point equidistant from A and B lies on the right bisector of AB.*

In $\triangle$s APM, BPM,

$$\left\{\begin{array}{ll} AM = MB, & \text{(By construction)} \\ AP = BP, & \text{(Given)} \\ PM \text{ is common.} & \end{array}\right.$$

$\therefore \triangle APM \equiv \triangle BPM,$ (Three sides of one $\triangle$, etc.)
and $\angle AMP = \angle BMP.$
$\therefore$ each is a right angle. (Equal supp. $\angle$s)
$\therefore XY$ is the right bisector of AB.
$\therefore P$ lies on the right bisector of AB.
$\therefore$ any point which is equidistant from A and B is on the right bisector of AB.

(2) *To prove that any point on the right bisector of AB is equidistant from A and B.*

Let Q be any other point on XY.
Join QA, QB.
In $\triangle$s AQM, BQM,

$$\begin{cases} AM = MB, \\ QM \text{ is common}, \\ \angle AMQ = \angle BMQ. \end{cases}$$

$\therefore \triangle AQM \equiv \triangle BQM,$ (Two sides and included $\angle$)
and $QA = QB$.
That is, Q is equidistant from A and B.
$\therefore$ any point on the right bisector of AB is equidistant from A and B.

Conclusion from (1) *and* (2):

XY, the right bisector of AB, is the locus of P.

Discussion of Proposition 4. You have noted, no doubt, that there were two steps (1) and (2) in the proof. In (1), it was shown that any point satisfying the law lies on the right bisector. In (2), it was shown that any point on the right bisector satisfies the law. The final conclusion (that the right bisector is the locus) was derived from the two proofs.

Your attention is drawn to the fact that the conclusion of (2) is the converse of the conclusion of (1).

EXERCISE

A gardener wished to place a bird bath so as to be the same distance from three trees not in the same straight line. After

some trial and error, he found the correct location. If he had been familiar with Proposition 4, he might have used another method. Using Proposition 4, how would you solve the problem?

* * *

Let A, B and C be three points representing the locations of the trees. Since the bird bath must be equidistant from A and B, on what line must it lie? * * * Draw the line. Since the bath must be equidistant from B and C, on what line must it lie? * * * Draw it. What point is equidistant from A, B and C? Why?

Let us state the gardener's problem in the precise language of geometry.

PROPOSITION 5. PROBLEM

To find a point equidistant from three fixed points not in the same straight line.

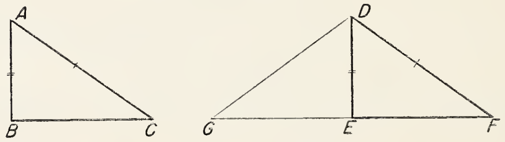
Write a formal proof for the solution of Proposition 5.

* * *

Before we continue with our work on loci, it is necessary to prove a fourth theorem dealing with the congruence of triangles.

PROPOSITION 6. THEOREM

If the hypotenuse and one other side of a right triangle are respectively equal to the hypotenuse and one other side of another right triangle, the two triangles are congruent.



Given: ABC and DEF are triangles right-angled at B and E and such that $AC = DF$, and $AB = DE$.

Required: To prove that $\triangle ABC \equiv \triangle DEF$.

Proof: Apply $\triangle ABC$ to $\triangle DEF$ so that A falls on D and AB falls along DE.

Since $AB = DE$,

B falls on E.

Let C fall at G on the side of DE remote from F.

$\angle DEG = 90°$. (Given)

$\angle DEF = 90°$. (Given)

$\therefore\ \angle DEG + \angle DEF = 180°$.

$\therefore\ GEF$ is a straight line.

Since $DG = DF$, (Given)

$\angle F = \angle G$, (Angles opp. equal sides)

$= \angle C$.

In $\triangle$s ABC, DEF,

$$\begin{cases} \angle B = \angle E, & \text{(Each is 90°)} \\ \angle C = \angle F, & \text{(Proved)} \\ AB = DE. & \text{(Given)} \end{cases}$$

$\therefore\ \triangle ABC \equiv \triangle DEF$. (Two $\angle$s and a side)

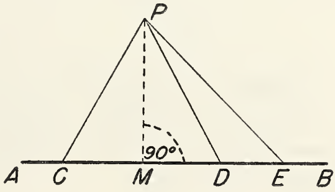
EXERCISES

1. In Proposition 6, why was it necessary to prove that GEF is a straight line?

2. Give the enunciations of four theorems dealing with the congruence of triangles.

Definition: ***The distance from a point to a straight line is the length of the perpendicular from the point to the line.***

Thus, the distance from P to AB is PM where $\angle PMB = 90°$. No matter how many other lines, PC, PD, PE, etc. you draw from P to AB, you will find that the perpendicular PM is the



shortest. The fact that the perpendicular is the shortest line from a point to a given line cannot be proved. You are asked to regard the statement as an assumption.

3. If a point is equidistant from the arms of an angle, it lies on the bisector of that angle.

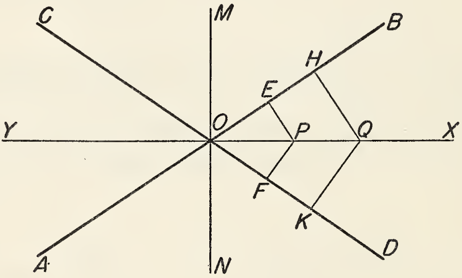
4. AB and CD are two straight lines intersecting at O. OX is the bisector of $\angle BOD$. XO is produced to Y. Prove that OY is the bisector of $\angle COA$.

5. AB and CD are two straight lines intersecting at at OX is the bisector of the $\angle BOD$ and OY is the bisector of $\angle COA$. Prove that XOY is a straight line.

6. AB and CD are two straight lines intersecting at O. YX bisects $\measuredangle$ COA and BOD. MN bisects $\measuredangle COB$ and AOD. Show that $MN \perp YX$.

PROPOSITION 7. THEOREM

If a point moves so that it is equidistant from two given intersecting straight lines, its locus is the pair of lines which bisect the angles between the given lines.



Given: AB and CD are two straight lines, intersecting at O.

Required: To prove that the locus of points equidistant from AB and CD is the pair of lines which bisect the angles between AB and CD.

(1) *To prove that any point equidistant from AB and CD lies on one or other of the pair of lines which bisect the angles between AB and CD.*

Proof: In $\angle BOD$, take a point P which is equidistant from AB and CD.

Join OP. Produce OP to X. Draw $PE \perp AB$ and $PF \perp CD$.

In $\triangle$s OPE, OPF,

$$\begin{cases} PE = PF, & \text{(Given)} \\ OP \text{ is common,} & \\ \measuredangle PEO \text{ and } PFO \text{ are right angles.} & \text{(Construction)} \end{cases}$$

$\therefore \triangle OPE \equiv \triangle OPF$, (Hypot. and one side)

and $\angle EOP = \angle FOP$.

$\therefore P$ lies on the bisector of $\angle BOD$.

Similarly, it may be proved that P may lie (*a*) on OY, the bisector of $\angle COA$, (*b*) on OM, the bisector of $\angle COE$, or (*c*) on ON, the bisector of $\angle AOD$.

But, XO and OY are in the same straight line, and MO and ON are in the same straight line. (Ex. 5, p. 386.)

$\therefore$ any point which is equidistant from AB and CD lies on one or other of the pair of lines which bisect the angles between the lines.

(2) *To prove that any point on the bisectors of the angles between AB and CD is equidistant from AB and CD.*

The point O on YX is the same distance (namely, zero) from AB and CD.

Now take any point, other than O, on YX, the bisector of $\measuredangle COA$, BOD. Let the point be Q. Draw $QH \perp OB$ and $QK \perp OD$.

In $\triangle$s OQH, OQK,

$$\begin{cases} OQ \text{ is common,} & \\ \angle QOH = \angle QOK, & \text{(Given)} \\ \angle QHO = \angle QKO. & \text{(Each is 90° by const.)} \end{cases}$$

$\therefore \triangle OQH \equiv \angle OQK$,

and $QH = QK$.

That is, Q is equidistant from AB and CD.

Similarly, it may be shown that any point on MN, the bisector of $\measuredangle$ COB and AOD, is equidistant from AB and CD.

Conclusion from (1) *and* (2):

Hence, the locus of a point which moves so that it is equidistant from the intersecting straight lines AB and CD consists of the two straight lines which bisect the angles between AB and CD.

EXERCISES

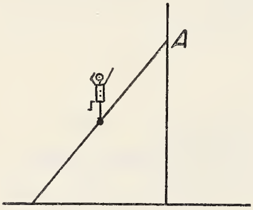
1. Why was it necessary to prove Proposition 6 before proving Proposition 7?

2. A rectangle $ABCD$ stands vertically on the base AB, If it turns round BC as axis, what is the locus of (*a*) A, (*b*) DC. (*c*) AD? Formal proofs need not be given. (The loci are not necessarily confined to a plane.)

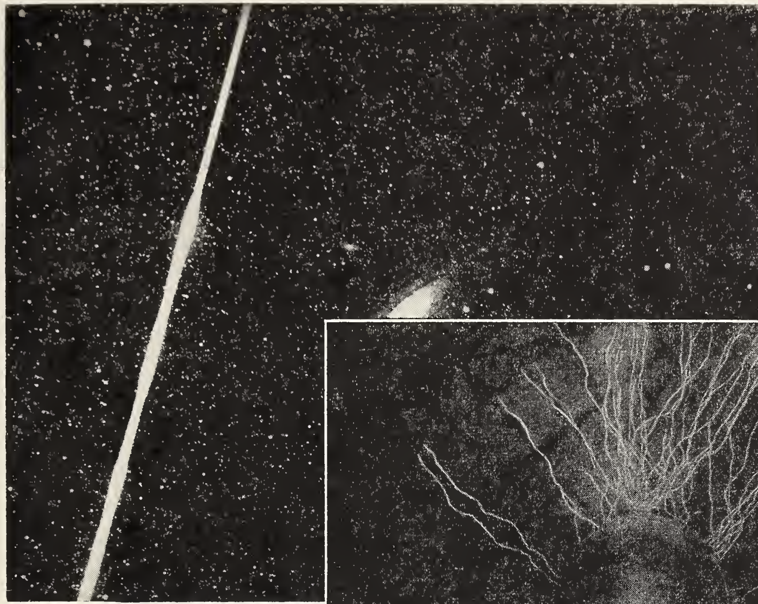
3. What is the locus of the vertices of the triangles which have a common base and in which the medians drawn to the base are equal to a given straight line? Do not give a proof.

4. What is the locus of the points of intersection of the diagonals of all rectangles which have one side fixed in length and in position? A proof need not be given.

5. Find the locus of the vertices of triangles equal in area, and standing on the same base and on the same side of it. Give the complete proof.



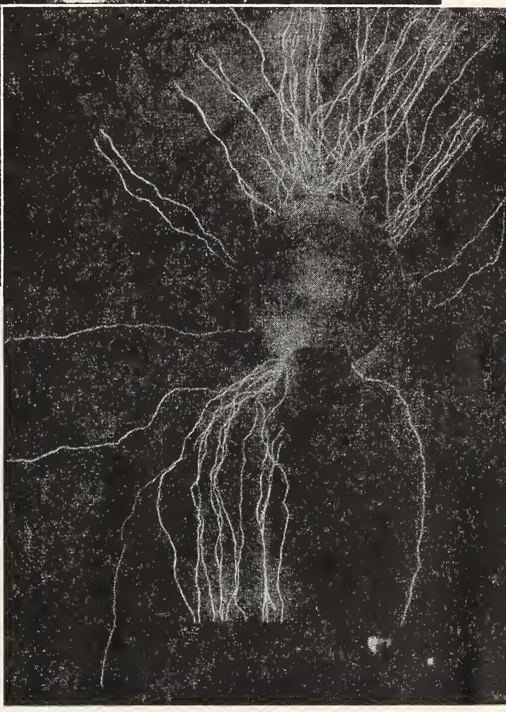
6. A man was half-way up a ladder which was leaning against the wall of a house. The ladder slipped so that a point at the end A followed a vertical line on the wall, and the man remained on the same rung throughout the fall. Using instruments, plot the locus of a speck of mud on the toe of his shoe.



(i)

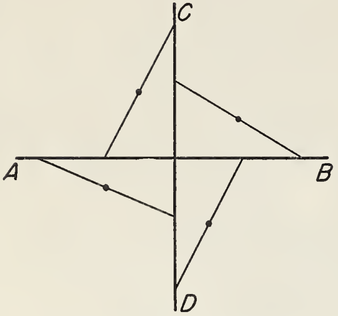
Courtesy Joseph Klepesta, American Museum of Natural History, New York

Interesting loci in nature: (i) the path of a fire-ball in space; (ii) the paths of electrons at the terminal of a 2,000,000-volt electrostatic generator.



(ii)

Courtesy News Service, Massachusetts Institute of Technology



7. A line of fixed length moves so that its ends are always on two perpendicular straight lines AB and CD. Using instruments, plot the locus of the middle point of the moving line. (See No. 7 under the heading, "When are two straight lines equal?" (page 345.)

In 1927, Mr. G. R. Smith, formerly a member of the staff of the Ottawa Normal School, produced a moving picture to illustrate this locus.

8. (*a*) ABC is a triangle. D and E are the midpoints of AB and AC respectively. Prove $DE \parallel BC$. (*Hint:* Through C draw $CF \parallel BD$ to meet DE produced at F.)

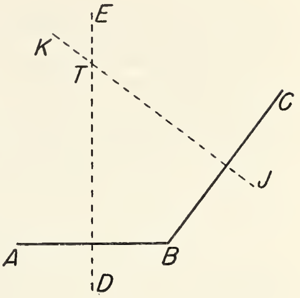
(*b*) AB is a straight line fixed in position, but not in length. P is a fixed point. In your opinion, what is the path on which the midpoints of straight lines drawn from P to AB lie? Give evidence in support of your opinion, but do not give a formal proof.

9. Using squared paper, plot the locus of a point (x, y) which moves in accordance with the law (*a*) $y = 2x$; (*b*) $y = 3x - 2$; (*c*) $2x + y = 4$. (NOTE: The equation is an algebraic statement of the law obeyed by the point.)

10. Plot the locus of a point (x, y) which moves in accordance with the law $y = x^2$.

The Intersection of Loci

In solving the gardener's problem (p.383), you drew the locus DE of points equidistant from the fixed points A and B. Then, you drew the locus JK of points equidistant from the fixed points B and C. Hence $TA = TB$ and $TB = TC$. That is, $TA = TB = TC$. In other words, T is equidistant from A, B, and C.



This exercise provides an example of the solution of a problem by finding the point or points of intersection of two loci. Let us consider another simple example:

A and B are two fixed points 1.5 *inches apart. Find a point C which is* 2 *inches from A and* 3 *inches from B.*

Mark the points A and B on your paper. What is the locus of points 2 inches from A? * * * Draw the locus. * * * What is the locus of points 3 inches from B? * * * Draw it. Obviously, each of the two points of intersection is 2 inches from A and 3 inches from B. Recall the construction of a triangle when given the lengths of the sides.

Now try the following exercises.

EXERCISES

1. A point (x, y) moves in accordance with the law $2x + 3y = 11$. A second point (x, y) moves in accordance with the law $x - 2y = -5$. Plot the loci on squared paper. Is there a point which obeys both laws? Give the location of the point. Verify your answer by solving the equations.

2. The captain of a ship found that his position at sea was 40° North Latitude and 20° West Longitude. Show that he

described his position by giving the point of intersection of two loci.

3. (*a*) AB is a straight line at a distance of 2 cm. from a point O. Find a point on AB which is 3 cm. from O.

(*b*) How many solutions are there for Exercise 3 (*a*)?

(*c*) How many solutions has the following problem? AB is a straight line at a distance of 2 cm. from a point O. Find a point on AB which is 2 cm. from O.

(*d*) How many solutions has the following problem? AB is a straight line at a distance of 4 cm. from a point O. Find a point on AB which is 3.5 cm. from O.

4. (*a*) In a given straight line find a point which is equidistant from two given intersecting straight lines.

(*b*) Under what conditions will there be (i) two solutions, (ii) one solution, (iii) an indefinite number of solutions to Problem 4 (*a*)?

5. Two straight lines AB and CD intersect each other at an angle of 60°. Find all the points that are 2.8 cm. from AB and 3.1 cm. from CD.

6. ***ABC is a triangle. Find a point which is equidistant from the three vertices.*** (See Prop. 5.)

7. ***ABC is a triangle. Find a point inside the triangle which is equidistant from the three sides.***

The Necessary and the Sufficient Condition

(Optional)

Consider the statement, *If the street is icy, it is slippery.* You would agree that the existence of ice warrants a belief that the street is slippery. This is expressed by saying that iciness is a ***sufficient condition*** for slipperiness.

Iciness, however, is not a ***necessary condition*** for slipperiness. Rain may produce a similar effect.

If a person is twenty-one years of age, he has satisfied only one of the requirements that must be fulfilled before he can vote.

Being twenty-one years of age or over is a necessary condition, but it is not a sufficient condition, for the right to vote.

If two squares have the side of the one equal to the side of the other, they are equal in area. It is sufficient to have the side of the one equal to a side of the other to make the squares equal in area. Also, it is necessary to have a side of the one equal to a side of the other if the two squares are to be equal in area. Hence, the condition 'Having a side of the one equal to a side of the other' is both ***sufficient and necessary*** for the equality of the two squares.

A is a ***sufficient condition*** of B if you are justified in asserting: *If A is true, then B is true.*

A is a ***necessary condition*** of B if you are justified in saying: *If B is true, then A is true.*

A is both ***a sufficient and a necessary condition*** of B if you can assert: *If A is true, then B is true* and *If B is true, then A is true.* The latter two statements may be brought together in the one statement: *If, and only if, A is true, then B is true*, or *B is true if, and only if, A is true.*

For example, here are two statements: (A) *The fathers of John and George are brothers* and (B) *John and George are first cousins.* It is apparent that we can say *If the fathers are brothers, the sons are first cousins.* Hence, A is a sufficient condition of B. It is equally clear that we cannot assert *If the sons are first cousins, the fathers are brothers.* Hence, A is not a necessary condition of B.

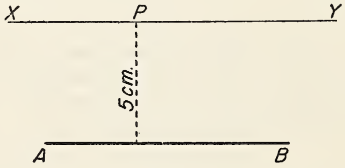
It is suggested that you give yourself some practice with the following pairs of statements. In each case, decide (*a*) if A is a sufficient condition for B; (*b*) if A is a necessary condition for B; (*c*) if A is both a sufficient and a necessary condition for B; (*d*) if A is neither sufficient nor necessary for B.

	A	*B*
1.	The shoes that Valene is wearing are much too small for her feet.	Valene's feet are sore.

2.	He is entitled to respect.	He is old and kindly.
3.	My watch has stopped.	The main spring is broken.
4.	The last digit of a number is 6.	The number is exactly divisible by 2.
5.	His birthday comes only once in four years.	He was born on February 29th.
6.	Alura is a born actress.	Alura's grandmother died at the age of ninety-seven.
7.	The seed germinated.	It was provided with moisture.
8.	Each of $\angle$s A and B is 90°.	The $\angle$s A and B are equal.
9.	$ABCD$ has a pair of adjacent sides equal.	$ABCD$ is a square.
10.	The two triangles have the three sides of the one respectively equal to the three sides of the other.	The angles of the one triangle are respectively equal to the angles of the other.
11.	$ab = 6$.	$a = 3$; $b = 2$.
12.	$x = 3$.	$4x - 12 = 0$.

Now let us glance backward at the subject of locus. Ex. 2 (a) on page 377 is as follows:

What is the locus of a point which moves so that it is always 5 centimetres from a given straight line and on one side of the line?



After examining the figure, you probably said to yourself, "It looks as if the locus of the point P is the line XY parallel to, and 5 centimetres distant from, the given line AB".

However, we pointed out that, before the supposed locus XY could be regarded with certainty as the true locus, it was necessary to prove two statements:

1. If any point obeys the law it must lie on XY.
2. If any point lies on XY, it must obey the law.

You will notice that:

(*a*) a proof of (1) is the same as showing that 'location on XY' is a *necessary condition* for 'obedience to the law';

(*b*) a proof of (2) is the same as showing that 'location on XY' is a *sufficient condition* for 'obedience to the law';

(*c*) each of (1) and (2) is the converse of the other.

In the formal proof that a supposed locus is the true one, both the necessary and sufficient conditions must be established, as in Propositions 4 and 7.

> *Illogical people often confuse necessary and sufficient conditions in their thinking.*

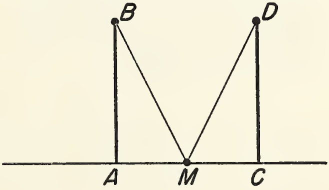
RECREATIONAL PROBLEMS

Intended only for Those with a Keen Interest in Locus

(*Proofs need not be given. The use of tracing paper in the transference of lengths will be found helpful.*)

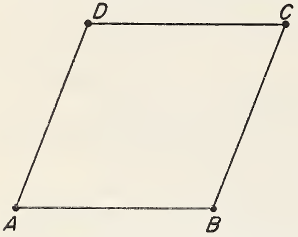
Linkage and Locus

1. AB and CD are equal vertical bars. $BM = DM =$ a constant length. MB and MD are free to move at B and D respectively. They are also linked at M. If AB and CD are moved simultaneously to the left and right, respectively, at the same speed, what is the locus of M?



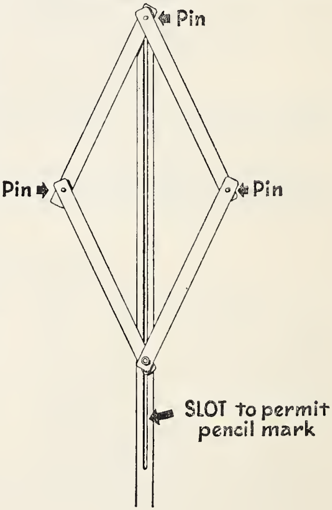
2. If CD is fixed in the position shown in the figure, and AB is moved to the left, what is the locus of M?

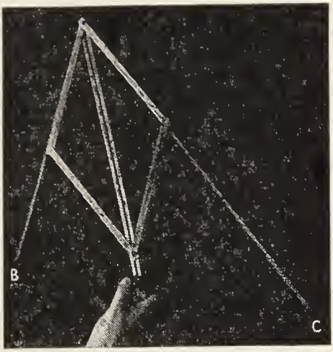
3. $ABCD$ is a rhombus consisting of four linked bars. If the position of AB is maintained, what is the locus of (*a*) the point C, (*b*) the point D, as the shape of $ABCD$ is altered?

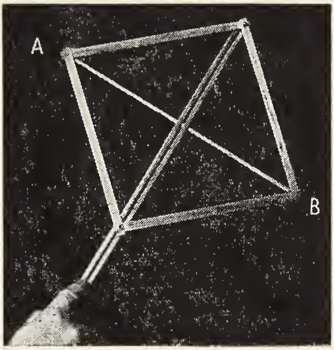


4. What is the locus of the intersection of the diagonals of the rhombus described in (3) as the shape is altered? (If you have difficulty with this problem, try it after you have studied the properties of circles in Chapter XVII.)

The topic of linkages is a thoroughly interesting one. You may recall the device used in the trisection of angles (Book I, p. 346). Another linkage, the pantograph, will be mentioned later in this book. The accompanying illustrations show a linkage which may be employed for (1) drawing the right bisector of a straight line; (2) bisecting an angle; (3) erecting a perpendicular to a straight line at a point on the line; (4) drawing a perpendicular to a straight line from a point not on the line. The following photographs show the linkage being used for the second and first of these purposes.





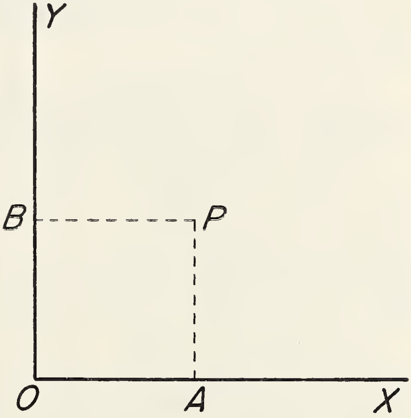


You may be interested in constructing linkages devised by yourself.

A familiar curve in mathematics and physics

5. OX and OY are two given perpendicular straight lines.

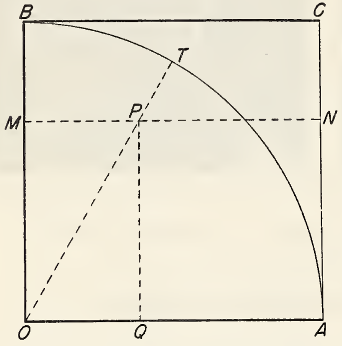
P is a point located one inch from OX and one inch from OY. Hence, the square $OAPB$ has an area of 1 sq. inch.



P moves in accordance with a rather curious law. Wherever P may be, the rectangle formed when perpendiculars are drawn from it to OX and OY is 1 sq. inch in area. Plot the locus of P.

The curve obtained is a ***hyperbola***. Perhaps you have already encountered it in connection with Boyle's Law. ($PV =$ constant.)

Hippias, a Greek, who lived about 450 years before the Christian era devised an ingenious method of trisecting angles.



6. OA turns through 90° at a uniform rate and reaches its final position OB.

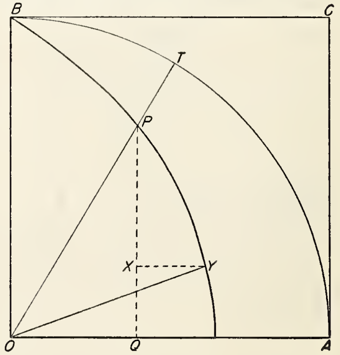
In the same time, a line perpendicular to OB moves at a uniform rate from its first position OA to its final position BC.

OT and MN are simultaneous positions of the moving lines. (OT has rotated through 60° which is two-thirds of 90°; MN, in the same time, has travelled two-thirds of the distance from OA to BC.)

OT and MN intersect at P.

Plot the locus of P. The curve is known as the ***Quadratrix of Hippias***. The figure on the right is a drawing of the curve.

To trisect any angle such as $\angle AOT$, it is necessary to find the third part of QP, namely, QX. (You may do this by calculation and measurement.) Then join OY.

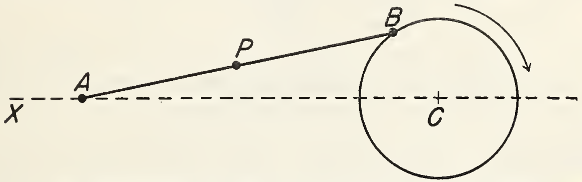


Any fraction of an angle may be found similarly. To obtain the eleventh part of $\angle AOT$, find the eleventh part of QP and complete the construction.

The quadratrix of Hippias may also be used for squaring the circle.

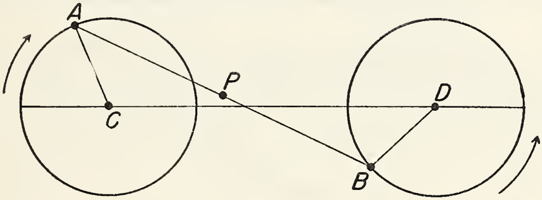
What a thrill Hippias must have had! If, instead of inventing a curve, he had carved his name on a desk, it is unlikely that he would be remembered for twenty-three centuries.

To plot the locus of a point on the connecting-rod of an engine.



7. AB is the connecting rod. The end B moves on the circle with centre C. The end A moves back and forth along the line XC. P is a given point on the rod. It is required to find the locus of P.

A more complicated locus.



8. AB is a rod connected by cranks AC and BD of equal length to the centres C and D. $AB = CD$. One crank rotates clockwise; the other, counter-clockwise. P is the midpoint of AB. Plot the locus of P.

The locus of a dizzy, but persistent fly.

9. A fly at the rim of a wheel began to walk at a uniform speed along a spoke. When the fly arrived at the stationary

centre, it began immediately to walk towards the rim along the continuation of the same spoke.

As the fly walked, the wheel rotated uniformly. When the fly reached the centre, the wheel had turned through 180°, and, when the fly arrived at the rim again, the wheel had completed one rotation.

Plot the locus of the fly during a complete rotation. Take the radius of the wheel as 3.6 centimetres.

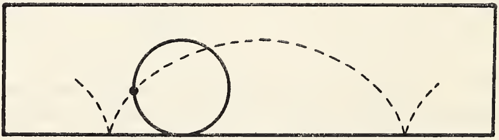
A dizzy fly without a future.

10. A second fly found itself at the centre of a wheel with a very large radius. As the wheel rotated uniformly, the fly walked steadily along a spoke. Plot the fly's locus.

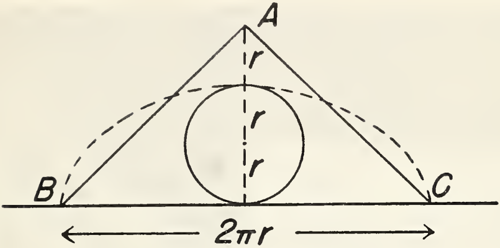
The curve you obtained is the well-known ***Spiral of Archimedes.*** You may wish to inform your friends that you have plotted the locus of a point moving with uniform velocity along a radius vector which is revolving about a fixed point with uniform angular velocity!

To plot the locus of a point on the circumference of a circle which rolls without slipping along a straight line.

11. Mark a point on the circumference of a fifty-cent piece and roll the coin along a straight line. What do you suppose is the locus of the point? * * * Make some rough drawings.



If you are at all interested in this problem, you will enjoy constructing a model and tracing the locus with its aid. The curve is called a ***cycloid.***



The curve has a number of interesting properties.

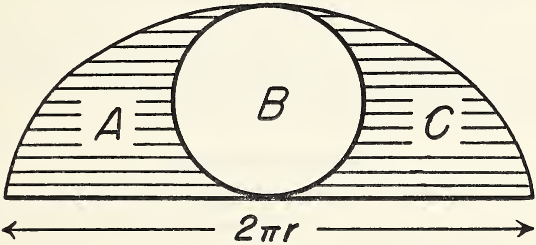
Area of $\triangle ABC = \frac{1}{2} \times 2\pi r \times 3r$

$= 3\pi r^2$

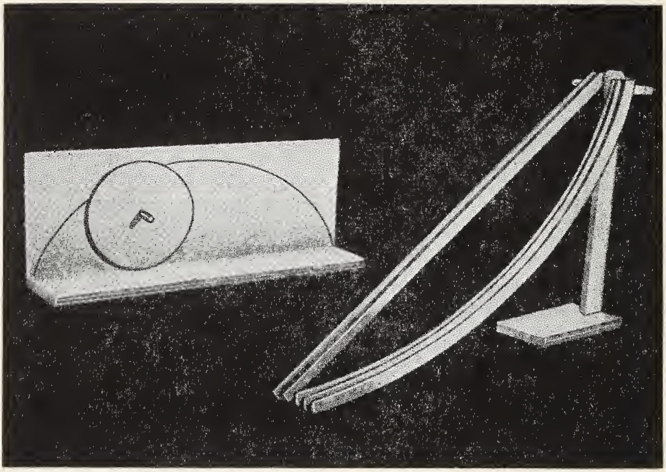
$= 3 \times$ area of the circle.

When the circle and curve are in the position below,

$$A = B = C = \pi r^2.$$



If two balls, one on a linear, the other on a cycloidal path, are released at the same moment, the ball on the cycloidal path reaches the bottom first. (See p. 402).



From the "Eighteenth Yearbook." Courtesy National Council of Teachers of Mathematics, Columbia University.

If two cycloidal paths are used, a ball released at the top of one will arrive at the bottom simultaneously with a ball released at the same time at the middle of the other.

REVIEW EXERCISES

A

1. With the aid of an example, explain what is meant by the phrase 'locus of a point'.

2. (*a*) A and B are fixed points. What is the locus of the vertices of isosceles triangles with base AB?

(*b*) What two statements must be proved before the supposed locus can be regarded as the true one?

(*c*) Give the two proofs.

3. What is the locus of a point which moves so that it is always 5 cm. from a line fixed in position, but not in length? A formal proof should be given.

4. Two points A and B are 3 inches apart. A variable point moves so that it is equidistant from A and B and not more than 3 inches from either. What is the locus of the point? A proof need not be given.

5. Two straight lines intersect at an angle of 45°. Find a point which is 2 cm. from one line and 3 cm. from the other. How many points satisfy the requirement?

6. XY is a line fixed in position, but not in length. A and B are two given points. Find a point P on XY such that $PA = PB$. Under what conditions would it be impossible to solve this problem?

B

7. (*a*) Give a *sufficient* condition for the equality of two angles based on the following statement: If two angles are vertically opposite each other, they are equal.

(*b*) Is the equality of two angles a *necessary* condition for being vertically opposite? Explain.

8. (*a*) Give a sufficient condition for the equality of two rectangles based on the following statement: If two rectangles have the same length and the same width, they are equal.

(*b*) Is the property of having the same length and the same width a necessary condition for the equality of two rectangles? Explain.

9. (*a*) What is the locus of a point which moves so that it is always 4 cm. from a straight line fixed in position, but not in length? (A proof need not be given.)

(*b*) What two statements must be proved before the supposed locus can be regarded as the true one?

(*c*) (i) Which proof demonstrates that 'location on the line' is a necessary condition for 'satisfaction of the requirement'?

(ii) Which proof demonstrates that 'location on the line' is a sufficient condition for 'satisfaction of the requirement'?

10. (*a*) A straight line BC, 8 cm. in length, is fixed in position. A point P moves so that $\triangle PBC$ has always an area of 12 sq. cm. What is the locus of P?

(*b*) On the line BC of (*a*) construct a $\triangle ABC$ given that the area is 12 sq. cm. and that $\angle BCA = 120°$.

11. A circle with radius one inch rolls around the outside of a square with side one inch. Find the locus of the centre of the circle. A proof need not be given.

C

12. $ABCD$ is a quadrilateral. Find a point P such that the $\triangle$s PAB and PDC are isosceles. Under what circumstances would there be (*a*) one solution; (*b*) no solution; (*c*) an infinitely large number of solutions?

13. Find the locus of the midpoints of all straight lines drawn from a given point to a given straight line. A formal proof need not be given.

14. Using your compasses, plot the locus of a point which moves so that the difference of its distances from two fixed points is one inch.

15. (*a*) The $\angle YOX = 90°$. What is the locus of a point which moves so that its distance from OX is less by 3 units than its distance from OY?

(*b*) Give an algebraic statement of the law which must be obeyed by the point.

16. (*a*) The $\angle YOX = 90°$. Find graphically the locus of a point which moves so that the sum of the square of its distance from OY and the square of its distance from OX is equal to 25 square units.

(*b*) Give an algebraic statement of the law which must be obeyed by the point.

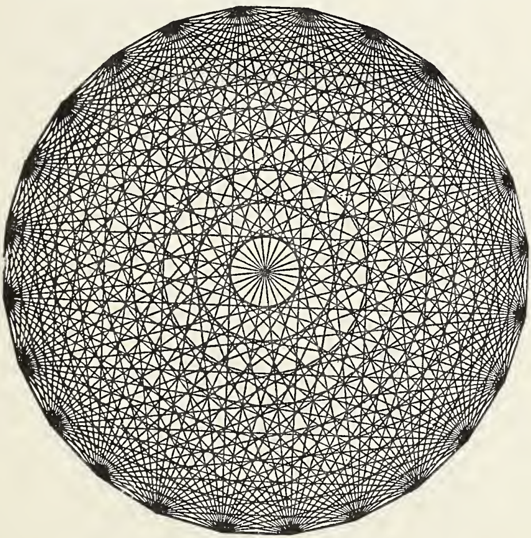
THE CIRCLE

In an age of amazing discoveries, it may astonish some to learn that the invention of the wheel is regarded as one of the greatest of all. Think for a moment of our plight if, by some miracle, the world were wheelless * * * The oldest wheel of which we know was made about fifty-five centuries ago and was found in Mesopotamia. There is no evidence of the use of the wheel in the New World before the arrival of the white man.

It would be surprising indeed if the properties of the circle had not been a subject of the closest investigation by mathematicians, both ancient and modern. We shall devote some time to a study of this interesting topic.

* * *

You will recall that a polygon was defined as a closed, broken straight line. If you draw a series of regular polygons with an ever-increasing number of sides, you will find that the shapes



Twenty-four-sided polygon with all diagonals

Reproduced from "Multi-sensory Aids in the Teaching of Mathematics", by kind permission of the National Council of Teachers of Mathematics and the Bureau of Publications, Columbia University, N.Y.

bear a closer and closer resemblance to a circle. The figure on p. 405, which is a regular polygon of merely twenty-four sides, illustrates this fact very well. However, you should know that no polygon can ever be a circle no matter how many millions of sides it may possess.

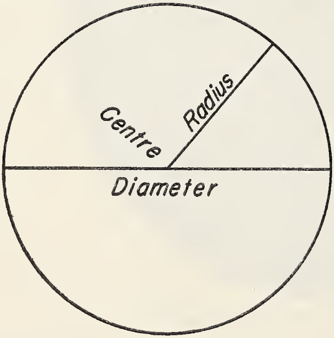
The ***circle*** is defined as a *closed, curved line every point on which is a constant distance from a given point.* With your knowledge of locus, you will have no difficulty in understanding that *a circle is the locus of a point which moves so that it is always the same distance from a fixed point.* The fixed point is, as you are aware, the ***centre*** of the circle.

You should not overlook the assumption contained in the statement that a circle is the locus of a point which moves so that it is always a constant distance from a fixed point. Every point on a ***sphere*** is at the same distance from the centre. The assumption in the definition of a circle is, of course, that the points are all in the same plane.

The same assumption was made in dealing with the locus of a point which moves so that it is equidistant from two fixed points. Had we not restricted the path of the moving point to one plane (in which the two fixed points lie), the locus would be, not the right bisector, but a plane through the right bisector and perpendicular to the line joining the two fixed points.

Since we are studying plane, not solid, geometry, attention will not be drawn again to this assumption.

The terms ***'circle'*** and ***'circumference'*** are often used interchangeably. When we have the perimeter (or length of the curved line) in mind, we use the word 'circumference'.



You will note that we defined the circle as a line. Frequently, the expression '*area of a circle*' is used. This must be regarded as a convenient short form of '*the area of surface bounded by a circle*'.

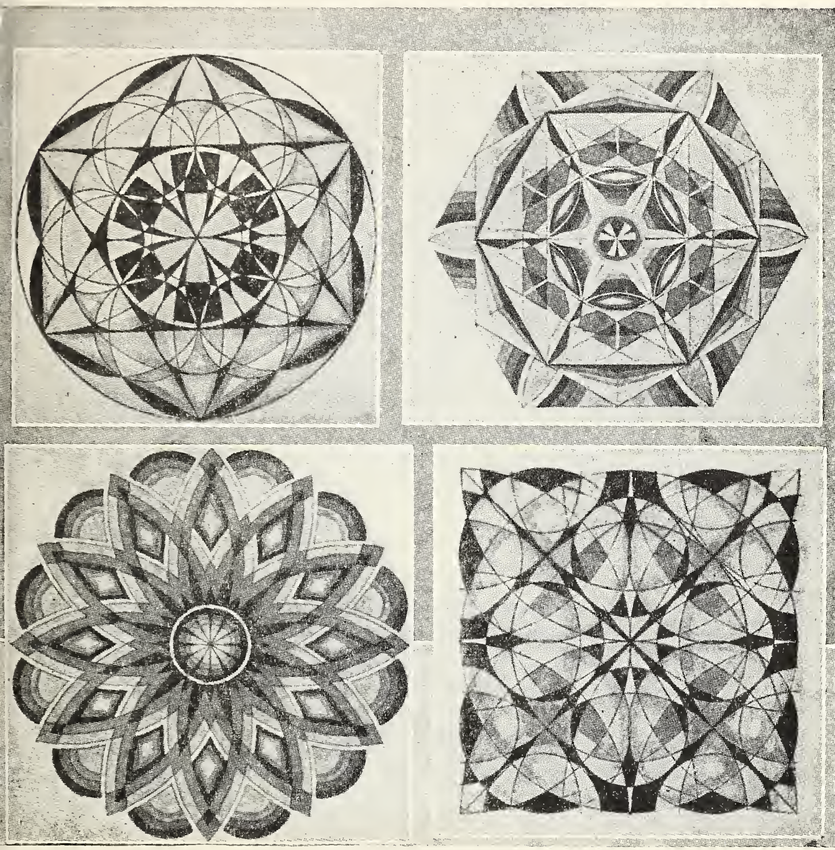
A ***radius*** is *a straight line drawn from the centre to the circle.*

A ***diameter*** is *a straight*

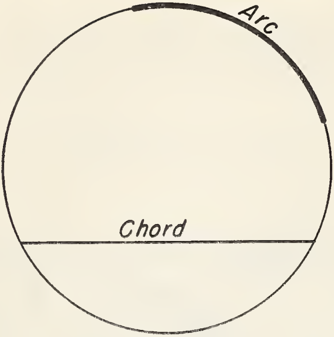
line drawn through the centre and having its ends on the circle. Hence, a diameter has twice the length of a radius.

From the definition of a circle, we can at once make the following inference:

Circles of equal radii are equal in all respects. In other words, circles of equal radii have equal circumferences and enclose equal areas.



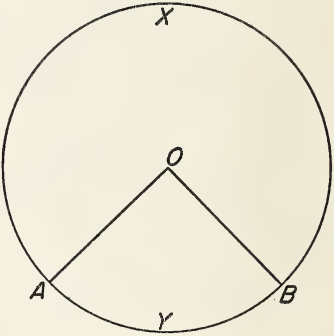
The circle is the basis of many interesting designs for jewellery, linoleums, etc. The four drawings were made by pupils.

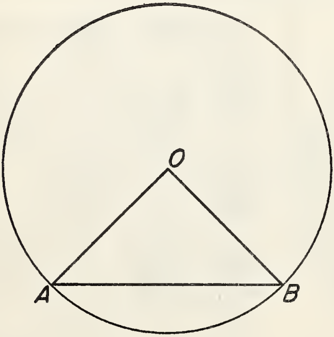


Any two points on a circle divide it into two parts each of which is called an ***arc*** *of the circle.*

A ***chord*** of a circle is *the straight line joining any two points on the circle.* What is the name of a chord on which the centre of the circle lies?

In the accompanying figure, there are two ***angles at the centre*** of the circle. One is the angle AOB *standing on*, or *subtended by*, the smaller (or minor) arc AYB. The other is the reflex angle AOB *standing on*, or *subtended by*, the major arc AXB.

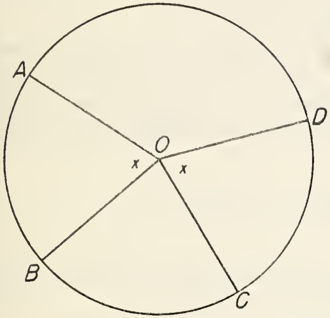




The two angles AOB may also be described as angles at the centre *standing on*, or *subtended by*, the chord AB.

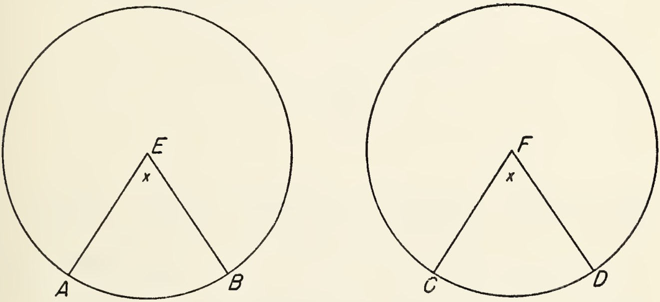
EXERCISES

1. ***If two angles at the centre of a circle are equal, they stand on equal arcs.***



(*Hint:* Fold the circle so that the radius OB falls along the radius OC.)

2. State and prove the converse of Ex. 1. (This converse and the others in this set of exercises are as important as the exercises of which they are converses.)



3. ***If an angle at the centre of a circle is equal to an angle at the centre of an equal circle, the arcs on which they stand are equal.*** (*Hint:* Apply the circle with centre E to the circle with centre F so that E falls on F, and so that EA lies along FC.)

4. State and prove the converse of Ex. 3.

5. ***If two chords in a circle are equal,*** (*a*) ***the angles sub-***

tended at the centre are equal; (b) the major and minor arcs cut off by one chord are respectively equal to the major and minor arcs cut off by the other.

6. State and prove the converse of Ex. 5 (a).

7. State and prove the converse of Ex. 5 (b).

8. ***If a chord in one circle is equal to a chord in an equal circle, (a) the angles subtended at the centres are equal; (b) the major and minor arcs cut off by one chord are respectively equal to the major and minor arcs cut off by the other.***

9. State and prove the converse of Ex. 8 (a).

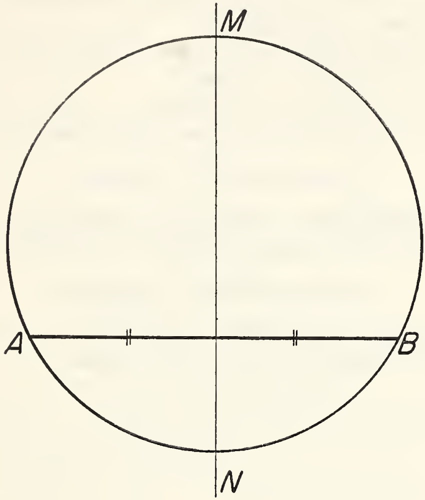
10. State and prove the converse of Ex. 8 (b).

* * *

The conclusions derived from the preceding ten exercises may be used without proof in subsequent exercises.

PROPOSITION 8. THEOREM

If a straight line bisects a chord of a circle at right angles, it passes through the centre of the circle.



Given: AB is any chord of the circle ABC. MN is the right bisector of AB.

Required: To prove that MN passes through the centre of the circle ABC.

Proof: MN is the right bisector of AB. (Given)

$\therefore$ MN is the locus of points equidistant from A and B.

But the centre of the circle is equidistant from A and B.

(Definition of circle)

$\therefore$ the centre lies on MN.

Discussion of Proposition 8. We shall have another look at the data and the conclusion:

Data	*Conclusion*
1. MN bisects the chord AB.	MN passes through the centre.
2. $MN \perp AB$.	

If you interchange datum (1) and the conclusion, what converse of Proposition 8 is obtained?

If you interchange datum (2) and the conclusion what converse of Proposition 8 is obtained?

These two converses are important. Their proofs will be left to you in the firm belief that you will find them easy exercises.

PROPOSITION 9. THEOREM

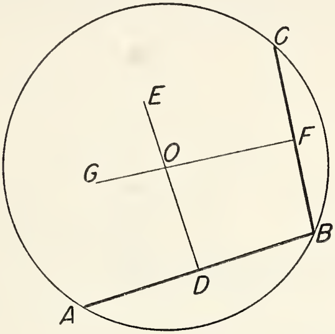
If a straight line through the centre of a circle is perpendicular to a chord, it bisects the chord.

PROPOSITION 10. THEOREM

If a straight line through the centre of a circle bisects a chord, it is perpendicular to the chord.

PROPOSITION 11. PROBLEM

To describe a circle through three points which are not in the same straight line.



Given: A, B and C are the given points.

Required: To describe a circle passing through A, B and C.

Construction: Join AB, BC.

Draw DE and FG, the right bisectors of AB and BC respectively, intersecting at O.

Then O is the required centre.

Proof: DE is the right bisector of AB. (Const.)
$\therefore$ the centre lies on DE. (Right bisector of chord)
FG is the right bisector of BC. (Const.)
$\therefore$ the centre lies on FG. (Right bisector of chord)
The only point common to DE and FG is O.
$\therefore$ O is the centre of the required circle.

With centre O and radius OA or OB or OC, describe the circle.

* * *

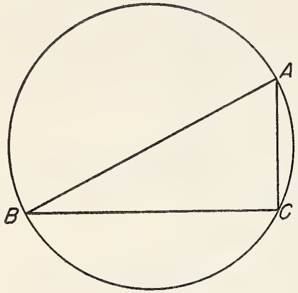
Now you should have no difficulty in solving the following problem:

To find the centre of a circle of which an arc is given.

* * *

DEFINITIONS: *If a circle passes through all the vertices of a polygon, it is said to be* ***circumscribed*** *about the polygon.* The circumscribed circle of a triangle is called its ***circumcircle.***

If the vertices of a polygon lie on a circle, the polygon is said to be ***inscribed*** *in the circle.*



The circle is circumscribed about the triangle ABC and is called the circumcircle. The triangle ABC is inscribed in the circle.

EXERCISES

1. ***Draw the circumcircle of a given triangle.***

2. (*a*) ABC is an equilateral triangle inscribed in a circle. Show that the minor arcs cut off by AB, BC and CA are equal.

(*b*) Prove that each of the three angles subtended at the centre is 120°.

3. Bisect a given arc of a circle and prove your construction.

4. Divide the circumference of a circle into six equal arcs. (Use compasses and straight edge.)

5. With the aid of your protractor, construct a regular pentagon.

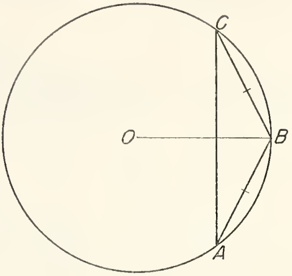
6. Find the locus of the middle points of parallel chords in a circle. A formal proof need not be given.

7. (*a*) Draw a circle of given radius to pass through two given points.

(*b*) Under what condition is it impossible to obtain a solution of Ex. 7(*a*)?

8. Through a given point P within a circle, draw a chord of which P is the middle point.

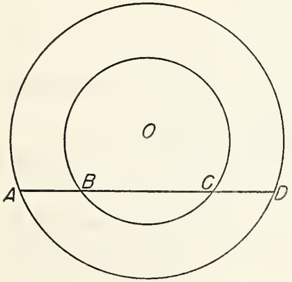
9. If O is the centre of the circle and $AB = BC$, prove (*a*) that OB bisects $\angle ABC$, (*b*) that OB is the right bisector of AC.



10. (*a*) A and B are two given points and XY is a straight line fixed in position but not in length. Draw a circle which will pass through A and B and have its centre on XY.

(*b*) Under what condition is it impossible to obtain a solution of Ex. 10 (*a*)?

11. ***If two circles intersect, the straight line joining their centres bisects the common chord at right angles.***



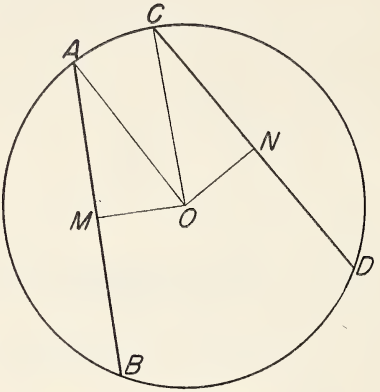
12. The two circles in the adjacent figure have a common centre O. Prove (*a*) that $AB=CD$; (*b*) that $\angle AOB = \angle DOC$.

13. ***If two chords of a circle are equidistant from the centre, they are equal.***

14. ***If two chords of a circle are equal, they are equidistant from the centre.***

PROPOSITION 12. THEOREM

If two chords of a circle are equidistant from the centre, they are equal.



Given: AB and CD are two chords of a circle with centre O. OM and ON are $\perp AB$ and CD respectively. $OM = ON$.

Required: To prove that $AB = CD$.

Proof: Join OA, OC.

In $\triangle$s OAM, OCN,

	$\left\{\begin{array}{r} OA = OC, \\ OM = ON, \\ \angle OMA = 90° = \angle ONC. \end{array}\right.$	(Radii of same circle) (Given) (Const.)
$\therefore$	$\triangle OAM \equiv \triangle OCN,$	(One side and hypot.)
and	$AM = CN.$	
But	$AB = 2\,AM$	(Chord is bisected by $\perp$ from centre.)
	$= 2\,CN$	(Chord is bisected by $\perp$ from centre.)
	$= CD.$	

Corollary: ***In equal circles, chords equidistant from the respective centres are equal.***

PROPOSITION 13. THEOREM

If two chords of a circle are equal, they are equidistant from the centre.

The proof is left as an exercise.

COROLLARY: ***In equal circles, equal chords are equidistant from the respective centres.***

EXERCISES

1. Chords equally distant from the centre of a circle subtend equal angles at the centre.

2. AB and AC are equal chords in a circle with centre O. Prove that OA bisects $\angle BAC$.

3. AB and CD are equal chords in a circle with centre O. AB and CD when produced through B and D respectively intersect at P. Prove that (*a*) OP bisects $\angle APC$; (*b*) $PB = PD$; (*c*) if PO is produced to meet AC at M, PM is the right bisector of AC; (*d*) $BD \parallel AC$.

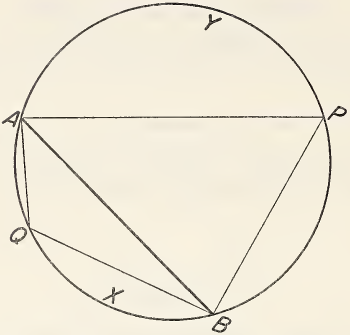
4. The major and minor arcs cut off by two chords equidistant from the centre, are respectively equal.

5. If two chords of a circle bisect each other, the point of intersection is the centre of the circle. (*Hint:* The indirect method of proof may be helpful here.)

6. Find the locus of the middle points of chords of a fixed length in a circle. A proof need not be given.

7. XY is a line fixed in position, but not in length. P is a given point not on XY. A circle with its centre on XY passes through P. Find a second point that must lie on the circle.

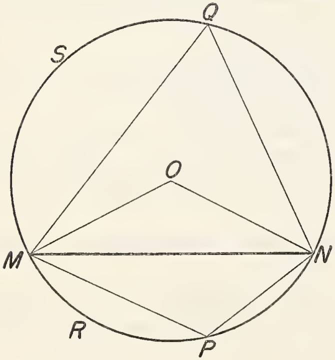
Angles at the circumference. In the following figure, AB is a



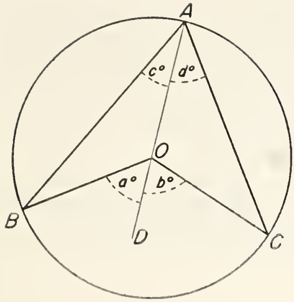
chord of the circle. P is a point on the circle. The $\angle APB$ is called ***an angle at the circumference.*** The $\angle APB$ may also be more particularly described as an angle at the circumference *standing on*, or *subtended by*, the minor arc AXB, or as an angle at the circumference *standing on*, or *subtended by*, the chord AB.

The $\angle AQB$ is another angle at the circumference. It stands on, or is subtended by, the major arc AYB. Also, it stands on, or is subtended by, the chord AB.

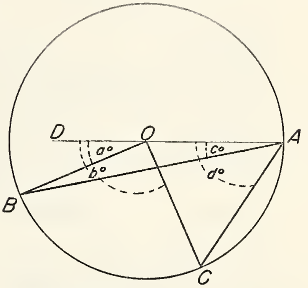
EXERCISES



1. With reference to the adjacent figure, describe in two ways each of the following angles:
   (*a*) $\angle MON$;
   (*b*) the reflex $\angle MON$;
   (*c*) $\angle MQN$;
   (*d*) $\angle MPN$.



2. O is the centre of the circle ABC. The number of degrees in certain angles is indicated. Prove that
 (a) $a = 2c$;
 (b) $b = 2d$;
 (c) $\angle BOC = 2\angle BAC$.



3. O is the centre of the circle ABC. The number of degrees in certain angles is indicated. Prove that
 (a) $a = 2c$;
 (b) $b = 2d$;
 (c) $b - a = 2(d - c)$, or that $\angle BOC = 2\angle BAC$.

PROPOSITION 14. THEOREM

If an angle is at the centre of a circle, it is double an angle at the circumference standing on the same arc.

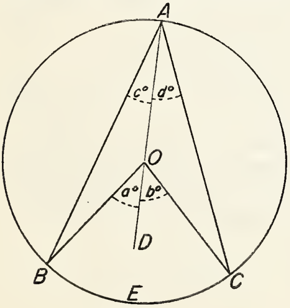


Fig. 1

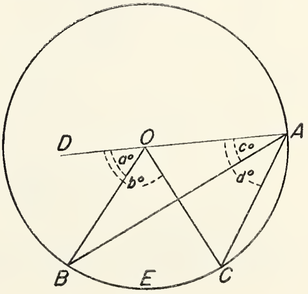


Fig. 2

Given: ABC is a circle with centre O. BOC is the angle at the centre, and BAC is an angle at the circumference standing on the same arc BEC. The number of degrees in certain angles is indicated in the figure.

Required: To prove that $\angle BOC = 2\angle BAC$.

Proof: Join AO and produce AO to D.

$AO = OB$. (Radii of same circle)

$\therefore\ \angle B = c°$. (Angles opp. equal sides)

$\angle BOD$ is an exterior angle of $\triangle ABO$.

$\therefore\ a° = \angle B + c°$.

That is, $a = 2c$.

Similarly, $b = 2d$.

Adding, in Fig. 1, we have

$$a + b = 2(c + d).$$

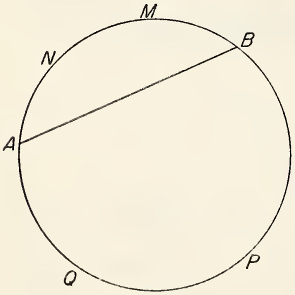
That is, $\angle BOC = 2\angle BAC$.

Subtracting, in Fig. 2, we have

$$b - a = 2(d - c).$$

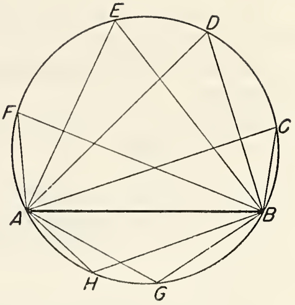
That is, $\angle BOC = 2\angle BAC$.

Definition: *If a chord is drawn in a circle, the figure which consists of the chord and the arc cut off by the chord is called a* ***segment of the circle.***



Two segments are illustrated by the adjacent figure. One is the segment $ABMN$. The other is the segment $ABPQ$.

If the chord AB had passed through the centre of the circle, each arc would have been a semicircle (or half circle), and the two segments would have been equal.

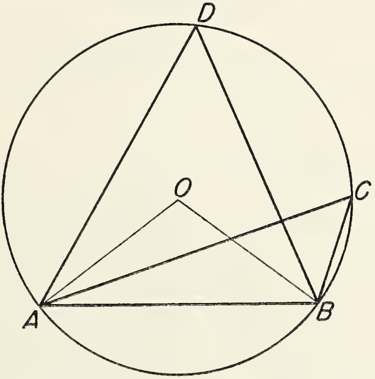


The angles ACB, ADB, AEB and AFB are *described as angles in the same segment* ($ABCDEF$) *of the circle.*

The angles AGB and AHB are also angles in the same segment ($ABGH$) of the circle.

PROPOSITION 15. THEOREM

Angles in the same segment of a circle are equal.



Given: ABC is a circle with centre O. ACB and ADB are angles in the same segment $ABCD$.

Required: To prove that $\angle ACB = \angle ADB$.

Proof: Join OA, OB.

$\angle C = \frac{1}{2} \angle AOB$. (Angle at circce. = $\frac{1}{2}$ angle at centre.)

$\angle D = \frac{1}{2} \angle AOB$. (Angle at circce. = $\frac{1}{2}$ angle at centre.)

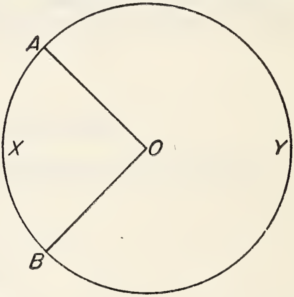
$\therefore \angle C = \angle D$.

The enunciation could have been stated in either of the following two ways:

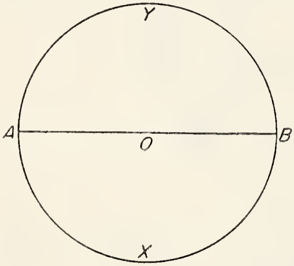
Angles at the circumference standing on the same arc are equal.

Angles at the circumference subtended by the same chord are equal.

A special angle at the centre and a special angle at the circumference. AOB is an angle at the centre of a circle, standing on the arc AXB.



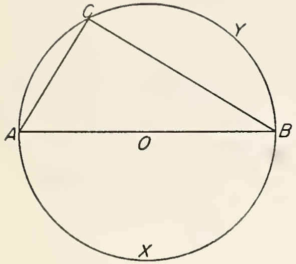
Suppose that OA is fixed in position and that OB is rotating in the counter-clockwise sense. Obviously, $\angle AOB$ is increasing and the arc AXB is increasing.



Let us suppose that OB takes the position shown in the second figure. OB is now in the same straight line as AO. What name is given to AB? How many degrees are contained in $\angle AOB$ at the centre standing on the arc AXB? How many degrees are in $\angle AOB$ standing on the arc AYB?

Clearly the straight $\angle AOB$ is the angle *at the centre* subtended by a semicircle.

Now let us consider an angle at the circumference when OB is the same straight line with the line AO.



The $\angle ACB$ at the circumference stands on the special chord AB called a diameter. Since AXB is a semicircle, $\angle ACB$ is an angle standing *on* a semicircle. Since AYB is also a semicircle, $\angle ACB$ is also an angle *in* a semicircle.

* * *

Let us glance again at the figure. Name an angle at the centre O * * * Name an angle at the circumference standing on the same arc. * * * How does $\angle ACB$ compare in size with $\angle AOB$? * * * What is the size of $\angle AOB$? * * * What then is the size of $\angle ACB$?

Using the conclusion of Proposition 15, we are now justified in saying that every angle in a semicircle is a right angle.

PROPOSITION 16. THEOREM

An angle in a semicircle is a right angle.

PROPOSITION 17. THEOREM

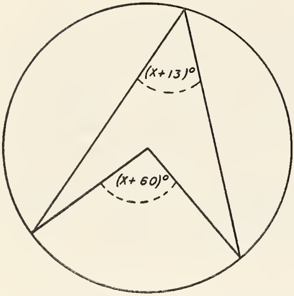
If an angle at the circumference is a right angle, it stands on a diameter of the circle.

The proof of this theorem is left as an exercise for the student. (*Hint:* Join to the centre each end of the arc on which the angle stands. Then, using the conclusion of Proposition 14, show that the angle between the two radii is a straight angle.)

EXERCISES

1. Show that, if two angles at the circumference stand on equal chords, they are equal.

2. Show that, if two angles at the circumference stand on equal arcs, they are equal.



3. The angle $(x + 60)°$ is at the centre of the circle. Find x and, hence, the size of the angle at the circumference.

4. Two circles with centres P and Q intersect at A and B. The diameters APC and AQD are drawn. Prove that CB and BD are in the same straight line.

5. A is a point on a semicircle of which BC is the diameter. The perpendicular from A to BC meets BC at D. Show that $\triangle$s ABC, ABD and ADC are equiangular.

6. ABC is a triangle in which $AB = AC$. Circles are described on AB and AC as diameters. Show that one of the points of intersection of the circles is the middle point of BC.

7. ABC is a triangle in which AB and AC are unequal. Prove that the circles with AB and AC as diameters intersect on BC or BC produced.

8. The median of a right triangle drawn to the hypotenuse is equal to half the hypotenuse.

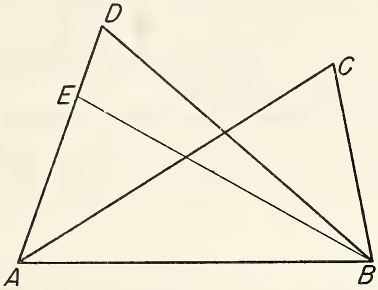
9. BC is a straight line fixed in length and position. A is a point which moves so that $\angle BAC = 90°$. Prove that the locus of A is a circle of which BC is the diameter.

* * *

Definition: *Four or more points through which a circle can be drawn are said to be* **concyclic.**

PROPOSITION 18. THEOREM

If the straight line joining two points subtends equal angles at two other points on the same side of it, the four points are concyclic.



Given: A and B are the given points. The equal angles ACB and ADB are subtended by AB at the points C and D, respectively, on the same side of AB.

Required: To prove that A, B, C, and D are concyclic.

Proof: If a circle is drawn through A, B, and C, it will either pass through D or it will cut AD or AD produced. Suppose that the circle does not pass through D, but cuts AD at E. Join BE.

$\angle C = \angle AEB.$ (Angles in same segment)

$\angle C = \angle ADB.$ (Given)

$\therefore \angle AEB = \angle ADB.$

That is to say, an exterior angle of $\triangle DEB$ is equal to an interior opposite angle.

This is impossible.

$\therefore$ the circle through A, B and C does not cut AD at E.

Similarly, it may be shown that the circle does not cut AD produced.

$\therefore$ the circle passes through D, and the points A, B, C, and D are concyclic.

Discussion of Proposition 18. It is clear that Proposition 18 is the converse of Proposition 15.

Proposition 18 provides us with another example of the use of indirect proof. There are only two possibilities: either the circle passes through D or it does not. If one of these possibilities can be ruled out, the other must be accepted.

In the proof, it was shown that, if we assume that the circle does not pass through D, an absurd situation arises. Hence, we must accept the other alternative.

In everyday life, indirect proof is used frequently. Sometimes wrong conclusions are drawn because all the possibilities are not taken into account. For example, if there are three possibilities and we consider only two of them, an incorrect conclusion *may* be obtained.

> *In the use of the indirect proof, whether in mathematics or in everyday life, no possibility should be overlooked.*

EXERCISES

1. Triangles ABC and ABD are drawn on the same side of the line AB. If the angles at C and D are both 57°, prove that A, B, C and D are concyclic.

2. Draw an equilateral triangle ABC, and construct its circumcircle. If D is a point on the same side of AB as C and if $\angle ADB = 60°$, show that D is on the circumcircle of the triangle ABC.

3. Use Ex. 2 in the construction of a triangle given the base AB, the vertical angle 60°, and the length of the median to the base.

4. In the discussion of Proposition 18, the following sentence occurred:

"If there are three possibilities and we consider only two of them, an incorrect conclusion *may* be obtained."

Explain why the word 'may' was used instead of 'will'.

5. From Propositions 18 and 15, an interesting corollary is derived:

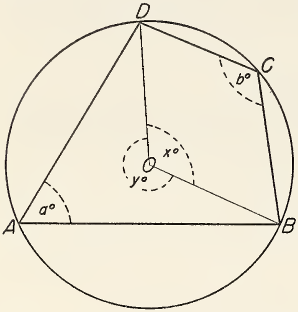
Corollary: ***The locus of the vertices of triangles standing on the same base and on the same side of it, with equal vertical angles, is an arc of a circle of which the base is a chord.***

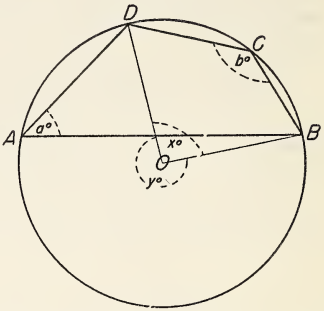
Give a formal proof of the corollary.

6. (*Optional*) State the contrapositive of the following assertion: If A, B, C, and D are not concyclic, the circle will cut BD at E.

PROPOSITION 19. THEOREM

If a quadrilateral is inscribed in a circle, the opposite angles are supplementary.





Given: $ABCD$ is a quadrilateral inscribed in a circle with centre O.

Required:

To prove that (1) $\angle DAB + \angle BCD = 2$ rt. angles,

(2) $\angle ABC + \angle CDA = 2$ rt. angles.

Proof: Join OB, OD.

$a = \frac{1}{2}x$. (Why?)

$b = \frac{1}{2}y$. (Why?)

$\therefore a + b = \frac{1}{2}(x + y)$

$= \frac{1}{2}$ of 360 (Why?)

$= 180$.

That is, $\angle DAB + \angle BCD = 180°$.

Similarly, it may be shown that $\angle ABC + \angle CDA = 2$ rt. angles.

DEFINITION: *A quadrilateral which can be inscribed in a circle is called a* **cyclic** (*or* **concyclic**) *quadrilateral.*

EXERCISES

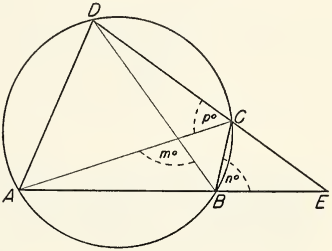
1. $ABCD$ is a cyclic quadrilateral. AB is produced to E. If the exterior $\angle CBE = 71°$, find the size of $\angle ADC$.

2. $ABCD$ is a cyclic quadrilateral. AD is produced to E. If the exterior $\angle CDE = n°$, find the size of $\angle ABC$.

3. ***If one side of a cyclic quadrilateral is produced, the exterior angle so formed is equal to the interior angle at the opposite vertex.***

4. In the adjacent figure, find the size of each of the following angles:

(a) $\angle ADC$; (b) $\angle ABC$; (c) $\angle ABD$; (d) $\angle CAB$; (e) $\angle CDB$; (f) $\angle DBC$; (g) $\angle DAC$; (h) $\angle ADB$; (i) $\angle ACB$; (j) $\angle BCE$; (k) $\angle DAB$; (l) $\angle BEC$.



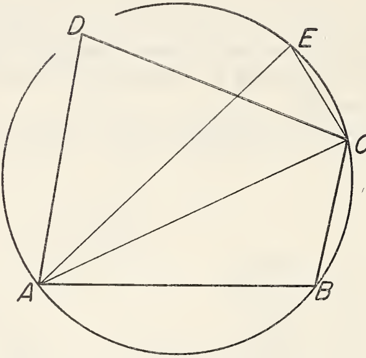
5. If a cyclic quadrilateral is a parallelogram, show (a) that it must be a rectangle; (b) that each diagonal is a diameter of the circle in which the quadrilateral is inscribed.

6. ABC is a triangle in which $AB = AC$. D and E are the midpoints of AB and AC respectively. Prove that $BCDE$ is a cyclic quadrilateral with equal diagonals.

7. State the converse of Proposition 19.

PROPOSITION 20. THEOREM

If two opposite angles of a quadrilateral are supplementary, its vertices are concyclic.



Given: $ABCD$ is a quadrilateral in which $\angle B + \angle D = 180°$.
Required: To prove that A, B, C, and D are concyclic.
Proof: Join AC.

Draw the circumcircle of $\triangle ABC$.

On the circle, take any point E on the same side of AC as D.

Join AE, CE.

Since $ABCE$ is a cyclic quadrilateral, (Const.)

$$\angle B + \angle E = 180°$$

$$= \angle B + \angle D. \quad \text{(Why?)}$$

$$\therefore \angle E = \angle D. \quad \text{(Why?)}$$

$\therefore A$, C, E, and D are concyclic. (Why?)

But the circle through A, C, and E passes through B. (Why?)

$\therefore A$, B, C, and D are concyclic.

EXERCISES

1. In Proposition 19, we are given a cyclic quadrilateral and required to prove that *both pairs* of opposite angles are supplementary. Why is it that in Proposition 20, the converse of

Proposition 19, we are given merely that two opposite angles are supplementary?

2. If an exterior angle of a quadrilateral is equal to the interior angle at the opposite vertex, the vertices of the quadrilateral are concyclic.

3. The circumcircle of an equilateral triangle is drawn. Prove that the angle in each segment outside the triangle is equal to 120°.

4. The circumcircle of a triangle is drawn. Prove that the sum of the angles in the segments outside the triangle is four right angles.

5. ABC is a triangle in which $AB = AC$. A parallel to BC cuts AB and AC at the points D and E respectively. Prove that (*a*) $BCED$ is a cyclic quadrilateral; (*b*) the diagonal $BE =$ the diagonal CD.

6. If a parallelogram is such that its vertices are concyclic, show that it is a rectangle.

7. $ABCD$ is a cyclic quadrilateral. AB and DC, when produced, meet at E. Show that $\triangle$s EAD and ECB are equiangular.

8. If the bisectors of the interior angles of a quadrilateral form a second quadrilateral, this latter quadrilateral is cyclic.

9. P, Q, R, S and T are successive concyclic points. If P and T are fixed, show that $\angle PQR + \angle RST$ is constant in magnitude.

10. ABC is any triangle. E and F are points on AC and AB respectively such that $BE \perp AC$ and $CF \perp AB$. BE and CF intersect at O. Prove that (*a*) B, C, E and F are concyclic; (*b*) BC is a diameter of the circle through B, C, E and F; (*c*) A, F, O and E are concyclic; (*d*) the centre of the circle through A, F, O and E is the midpoint of AO.

11. ABC is any triangle. E and F are points on AC and AB respectively such that $BE \perp AC$ and $CF \perp AB$. BE and CF intersect at O. AO is joined and produced to meet BC at D. Prove that $AD \perp BC$. (*Hint:* Join EF.)

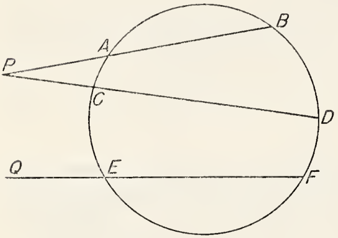
Ex. 11 may also be stated as follows: ***Show that the perpendiculars drawn from the vertices of a triangle to***

the opposite sides are concurrent. (Three or more lines are said to be ***concurrent*** when they pass through the same point.)

12. In the figure of Ex. 11 name six cyclic quadrilaterals. Join DE, EF and FD. This figure has a number of interesting properties. Try to discover some of them.

The point O is called the ***orthocentre*** of the triangle. The triangle DEF formed by joining the feet of the perpendiculars is called the ***pedal triangle.*** (Lat. *pes*, foot.)

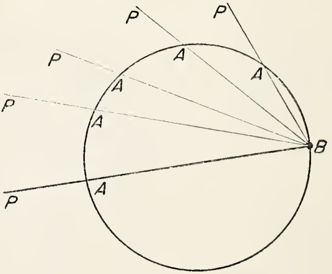
Tangents



A ***secant*** *is a straight line which cuts a circle in two points.*

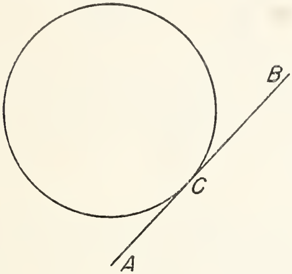
In the accompanying figure, three secants are shown, namely PAB, PCD and QEF.

Let us imagine that the secant PAB passes through a fixed point B on the circle and that BP is rotating in the clockwise sense.



As BP rotates, the point A will eventually come nearer and nearer, to B. When the point A arrives at B, the secant becomes a tangent to the circle. You may, then, regard a tangent as a limiting form of secant.

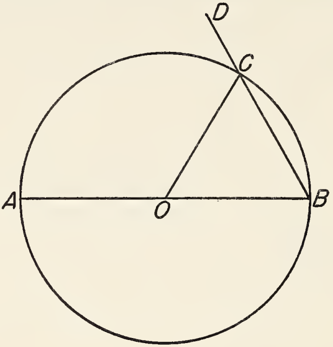
A ***tangent to a circle*** is defined as *a straight line which, however far it may be produced, meets the circle at a single point.* The word 'tangent' is derived from the Latin verb '*tangere*' meaning '*to touch*'. A tangent is said to ***touch*** the circle where they meet. The point of touching is called the ***point of contact.***



In the adjacent figure, AB is a tangent to the circle, and C is the point of contact.

PROPOSITION 21. THEOREM

If a straight line is drawn perpendicular to a diameter of a circle at one of the ends, it is a tangent to the circle.



Given: The straight line BD is $\perp AB$, a diameter of the circle with centre O.

Required: To prove that BD is a tangent to the circle.

Proof: If BD is not a tangent to the circle, then it must cut the circle at some point C. Join OC.

$OB = OC.$	(Radii of the same circle)
$\therefore \angle OCB = \angle OBC$	(Angles opp. equal sides)
$= 90°.$	($OB \perp BC$)

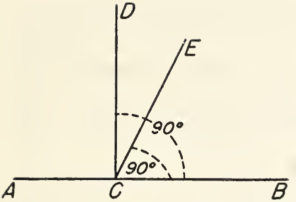
Hence, in $\triangle OBC$,

$$\angle O + \angle B + \angle OCB = \angle O + 180°.$$

This is impossible. (Sum of int. ∠s of $\triangle = 180°$.)

$\therefore$ BD cannot cut the circle.

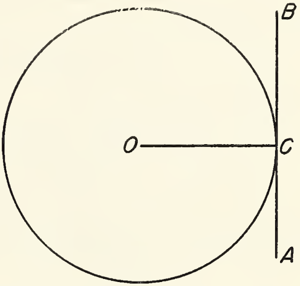
That is, BD is a tangent to the circle.



Can two different perpendiculars CD and CE be drawn to a line AB at a point C?

If this could happen, there would be two equal angles ECB and DCB. Thus, a part of an angle would be equal to the whole angle.

It is assumed that a part is not equal to the whole. Hence, there cannot be two lines CD and CE both perpendicular to AB at C.



We have seen (Prop. 21) that if $AB \perp$ radius OC at C, then AB is a tangent to the circle at the point C.

No other line can be drawn at $C \perp OC$. Hence, we make the following inference:

There can be only one tangent drawn to a circle at a given point on the circle.

Since there can be only one tangent drawn at C, and since the line $\perp OC$ at C is a tangent, we are justified in making a second inference:

If a straight line is a tangent to a circle, then the perpendicular to the tangent at the point of contact passes through the centre of the circle.

EXERCISES

1. Draw a tangent to a circle at a given point on the circle.

2. Prove that the radius perpendicular to a tangent is the right bisector of every chord parallel to the tangent.

3. To a given circle draw a tangent which is parallel to a given straight line.

4. To a given circle draw a tangent at right angles to a given straight line.

5. To a given circle draw a tangent which forms a given angle with a given straight line.

6. Two tangents to a circle with centre O intersect at the point P. Prove that (*a*) OP bisects the angle between the tangents; (*b*) the length of the two tangents from P to the points of contact are equal.

7. PM and PN are two tangents drawn to a circle with centre O. M and N are the points of contact. If $\angle MPN = n°$, find $\angle MON$.

8. P is a fixed point on a given straight line. Find the locus of the centres of circles which touch the given line at P. A formal proof need not be given.

9. Draw a circle of given radius to touch a given straight line at a given point. How many solutions has this problem?

10. Draw a circle which will touch a given straight line at a given point and have its centre on a straight line fixed in position, but not in length. Under what condition is it impossible to solve this problem? Under what condition are there an indefinite number of solutions?

11. Draw a circle which will touch a given straight line at a given point and pass through another given point. Under what condition is it impossible to solve the problem?

12. ***What is the locus of the centres of circles which touch two intersecting straight lines?*** Give a formal proof.

13. BA and BC are given straight lines. D is a fixed point on BC. Draw a circle which will touch BA and will touch BC at D.

14. Draw a circle which will touch two given intersecting straight lines and have its centre on a fixed straight line. How many solutions can you obtain? Is it ever impossible to solve the problem? Explain.

15. Draw a circle of given radius to touch two given intersecting straight lines? How many solutions can you find?

16. ***Describe a circle to touch three given straight lines.*** Consider the following cases:

(*a*) ***the given lines are parallel to each other;***

(*b*) ***only two of the given lines are parallel;***

(*c*) ***the three lines intersect.***

17. A circle of given radius moves so that it always touches a fixed straight line. Find the complete locus of the centre. (A formal proof need not be given.)

18. Describe a circle of given radius to touch a fixed straight line, and to have its centre on another fixed straight line. Under what condition will there be (*a*) two solutions; (*b*) no solution?

19. Describe a circle of given radius to pass through a given point and to touch a fixed straight line. Under what condition will there be (*a*) two solutions; (*b*) one solution; (*c*) no solution?

20. Draw a circle of given radius to touch a line fixed in position, but not in length, and so that its centre will be a given distance, greater than the radius, from a given point. Under what condition will there be (*a*) four solutions; (*b*) three solutions; (*c*) two solutions; (*d*) one solution; (*e*) no solution?

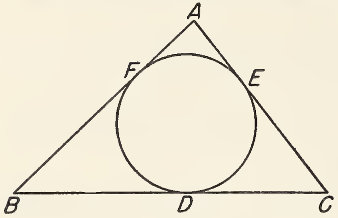
21. What is the locus of the centre of a circle which touches two given parallel straight lines? A formal proof need not be given.

22. Draw a circle which will touch two given parallel straight lines and pass through a given point. Under what condition will there be (*a*) two solutions; (*b*) one solution; (*c*) no solution?

23. Draw a circle which will touch two given straight lines and have its centre on a given straight line. Under what condition will there be (*a*) one solution; (*b*) no solution?

DEFINITION: *A circle is said to be* ***inscribed*** *in a triangle when it is within the triangle and touches the three sides.*

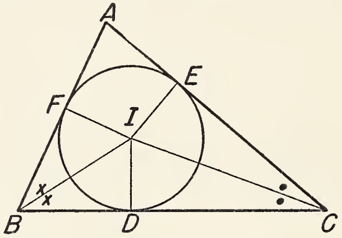
Thus, in the adjacent figure, the sides of the triangle AB, BC, and CA are tangents to the inscribed circle at the points F, D, and E respectively.



If you have solved Ex. 16 on page 437, you have already mastered the next proposition.

PROPOSITION 22. PROBLEM

To inscribe a circle in a triangle



Given: ABC is a triangle.

Required: To inscribe a circle in $\triangle ABC$.

Construction: Bisect $\measuredangle ABC$ and ACB. Let the bisectors meet at I.

Draw $ID \perp BC$.

With centre I and radius ID, draw a circle.

This is the required circle.

Proof: From I, draw $IE \perp CA$ and $IF \perp AB$.

Because BI is the bisector of $\angle ABC$,

$ID = IF$. (BI is locus of pts. equidistant from AB, BC.)

Similarly, $ID = IE$.

$\therefore ID = IE = IF$.

Hence, the circle with centre I and radius ID passes through D, E, and F.

$AB \perp$ radius IF. (Const.)

$\therefore$ AB is a tangent to the circle.

Similarly, BC and CA are tangents to the circle.

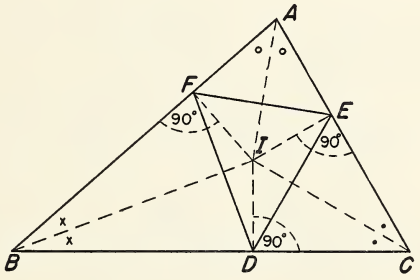
$\therefore$ circle DEF is the required inscribed circle.

EXERCISES

1. In the figure of Proposition 22, show that AI is the bisector of $\angle BAC$.

2. ***The bisectors of the interior angles of a triangle are concurrent.***

3. In the figure of Proposition 22, name three cyclic quadrilaterals.



4. I is the centre of the inscribed circle of $\triangle ABC$. IF, ID and $IE \perp AB$, BC and CA respectively. Prove that

(a) $\angle DFE = \frac{1}{2}(\angle A + \angle B)$;

(b) $\angle FDE = \frac{1}{2}(\angle B + \angle C)$;

(c) $\angle DEF = \frac{1}{2}(\angle C + \angle A)$;

(d) $\angle FID$ = exterior $\angle$ at B;

(e) $\angle DIE$ = exterior $\angle$ at C;

(f) $\angle EIF$ = exterior $\angle$ at A;

where A, B, and C are angles of $\triangle ABC$.

5. If I is the centre of the inscribed circle of an equilateral $\triangle ABC$, and ID, IE and IF are perpendicular to BC, CA and AB respectively, show that $\triangle DEF$ is equilateral.

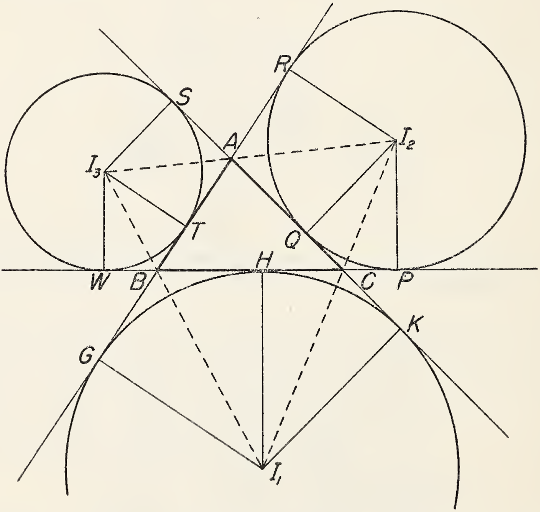
6. ABC is a triangle in which AB and AC are produced to M and N respectively. BI_1 is the bisector of $\angle CBM$ and CI_1 is the bisector of $\angle BCN$. $I_1G \perp AM$. $I_1H \perp BC$. $I_1K \perp AN$.

(*a*) Prove that $I_1G = I_1H = I_1K$.

(*b*) Prove that a circle with centre I_1 and radius I_1G touches AM at G, BC at H, and AC at K.

7. *Describe a circle to touch one side of a triangle internally and the other two sides externally.*

Definition: *A circle which touches one side of a triangle internally and the other two sides externally is an* **escribed circle** *of the triangle.*



In the figure drawn above, three escribed circles of the triangle ABC are shown.

Students who have done Ex. 6, p. 440 have learned how to draw the escribed circle touching the side BC and the sides AB and AC produced of the $\triangle ABC$.

Obviously, there are three solutions to Ex. 7, p. 440.

PROPOSITION 23. PROBLEM

To draw the escribed circles of a triangle.

The proof is left as an exercise.

EXERCISES

1. In the last figure, I_1C is the bisector of $\angle BCK$ and I_2C is the bisector of $\angle ACP$. Prove that I_1C and CI_2 are in the same straight line.

2. In the last figure, show that (*a*) AI_1 is the bisector of $\angle CAB$; (*b*) BI_2 is the bisector of $\angle ABC$; (*c*) CI_3 is the bisector of $\angle BCA$. This exercise may be re-worded as follows: ***The bisectors of two exterior angles of a triangle and the bisector of the third angle are concurrent.***

3. Show that AI_1, BI_2 and CI_3 pass through the centre of the inscribed circle of $\triangle ABC$.

4. Prove that (*a*) $AF = AE$; (*b*) $BF = BD$; (*c*) $CD = CE$; (*d*) $AS = AT$; (*e*) $BT = BW$; (*f*) $BH = BG$; (*g*) $CH = CK$; (*h*) $CQ = CP$; (*i*) $AQ = AR$. (See fig. on the next page.)

5. Let the measure of the length of BC be a; that of CA, b; and that of AB, c. (NOTE: a is measure of side opposite $\angle A$, etc.)

Let the measure of the perimeter of $\triangle ABC$ be $2s$.

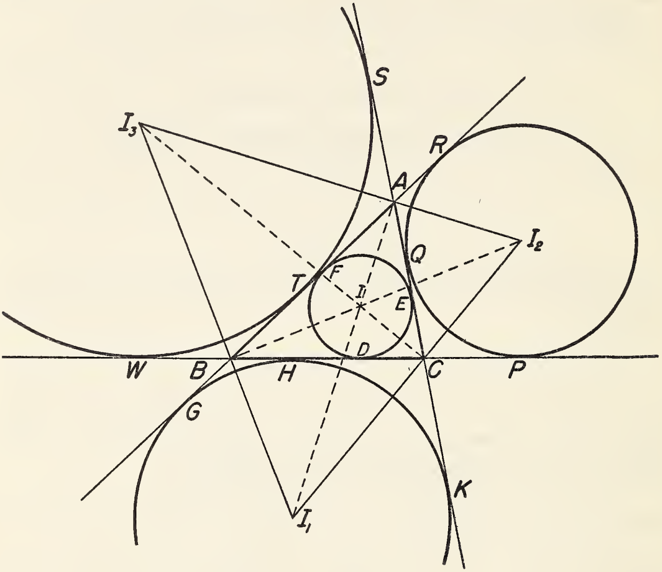
$$\therefore 2s = a + b + c,$$

$$\text{and } s = \tfrac{1}{2}(a + b + c).$$

Prove that (*a*) $AF = AE = s - a$; (*b*) $BD = BF = s - b$; (*c*) $CD = CE = s - c$; (*d*) $AG = AK = BP = BR = CS = CW = s$; (*e*) $BT = BW = s - a$; (*f*) $CH = CK = s - b$; (*g*) $AQ = AR = s - c$. (See fig. p. 442.)

6. Show that $rs = \triangle$, where $r =$ measure of the radius of the inscribed circle, $s =$ measure of the semi-perimeter of $\triangle ABC$,

$\triangle$ = measure of the area of $\triangle ABC$. (*Hint:* Note that $\triangle$ = sum of three small triangles.)



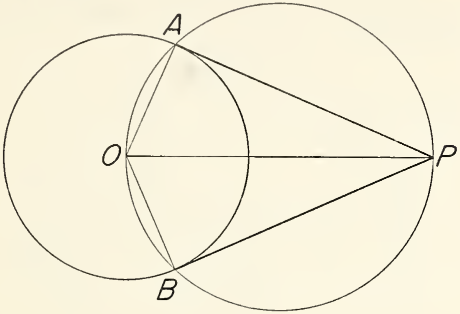
7. In the preceding figure, show that $\angle IBI_1 = \angle ICI_1 = 90°$.

8. ***P is a point outside a circle with centre O. PA and PB are tangents drawn from P to the circle, A and B being the points of contact. Show that the circle on OP as diameter passes through A and B.***

If you have solved the preceding exercise, you will have no difficulty with Proposition 24.

PROPOSITION 24. PROBLEM

To draw two tangents to a circle from a given point outside the circle.



Given: P is the given point outside the given circle with centre O.

Required: To draw two tangents to the circle from P.

Construction: Join OP.

On OP as diameter describe a circle. Let this circle intersect the given circle at the points A and B. Join PA and PB. Then PA and PB are the required tangents.

Proof: Join OA and OB.

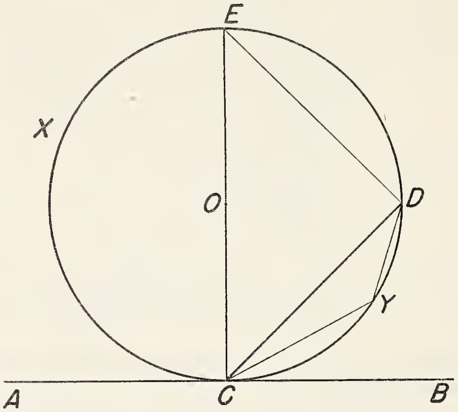
$\angle A = 90°$.	(Angle in a semicircle)
$\therefore$ PA is a tangent at A to the circle.	(OA is a radius.)
$\angle B = 90°$.	(Angle in a semicircle)
$\therefore$ PB is a tangent at B to the circle.	(OB is a radius.)

PROPOSITION 25. THEOREM

If two tangents are drawn to a circle from an external point, they are equal. (See Ex. 6, page 436.)

PROPOSITION 26. THEOREM

If from the point of contact of a tangent with a circle a chord is drawn, each angle between the chord and the tangent is equal to the angle in the segment on the other side of the chord.



Given: AB is a tangent at C to the circle with centre O.
CD is a chord drawn from the point of contact C.

Required: To prove that

(1) $\angle DCB =$ any angle in segment CXD,

(2) $\angle DCA =$ any angle in segment CYD.

Proof: Draw the diameter CE. Join DE.

Part (1)

$\angle BCE = 90°$. (Why?)

That is,

$\angle ECD + \angle BCD = 90°$

$= \angle ECD + \angle CED$. (Sum of inter. $\measuredangle$s of $\triangle = 180°$, and $\angle EDC$ in semicircle$=90°$.)

$\therefore \angle BCD = \angle CED.$

But, $\angle CED$ is an angle in the segment CXD.

$\therefore \angle BCD =$ any angle in the segment CXD.

Part (2)

Take any point Y in the segment CDY.

Join DY and YC.

$\angle DCB + \angle DCA = 180°$ (AB is a straight line.)

$= \angle CED + \angle CYD$. ($CYDE$ is a cyclic quad.)

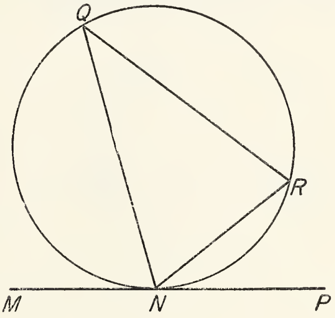
But $\angle DCB = \angle CED$, (Proved)

$\therefore \angle DCA = \angle CYD.$

But $\angle CYD$ is an angle in the segment CYD.

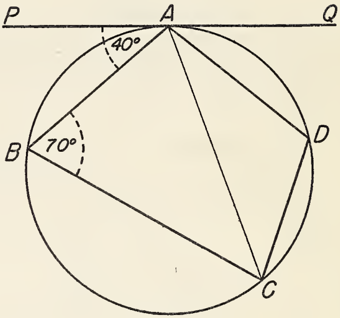
$\therefore \angle DCA =$ any angle in the segment CYD.

EXERCISES



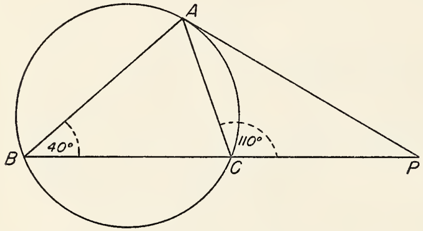
1. In the adjacent figure, MP is a tangent touching the circle at N. NQ is not a diameter. Prove that $\angle PNR = \angle NQR$.

2. Show that the acute angle between a tangent and a chord drawn from the point of contact is half the angle subtended by the chord at the centre.

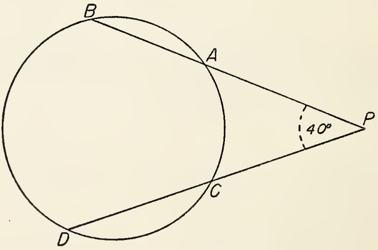


3. Find the size of each of the following angles in the adjoining figure:
   (a) $\angle ACB$;
   (b) $\angle BAC$;
   (c) $\angle ADC$.

4. PA is a tangent to a circle, the point of contact being A. PCB is a secant, cutting the circle at C and B. Show that $\triangle PAC$ and $\triangle PBA$ are equiangular.



5. PA is a tangent at A. Find the size of each of the following angles:
   (a) $\angle PAC$;
   (b) $\angle BAC$;
   (c) $\angle PAB$;
   (d) $\angle APC$.



6. Given that the angle in the arc $BACD$ standing on BD is 70°, find the size of the angle in the arc $ABDC$ standing on AC.

7. PQR is a triangle inscribed in a circle. If the tangent which touches the circle at P is parallel to QR, show that $\triangle PQR$ is isosceles.

8. ABC is a triangle inscribed in a circle. PA is a tangent to the circle at A. If AB is the bisector of $\angle CAP$, prove that $\triangle ABC$ is isosceles.

9. (*a*) PA is a tangent to a circle, the point of contact being A. AB is a chord such that $\angle PAB = 30°$. What is the size of the angle in (i) the major arc, (ii) the minor arc?

(*b*) How would you construct on a line AB, 2 inches in length, an arc of a circle at any point of which AB subtends (i) 30°, (ii) 150°? Prove that your construction is the correct one.

10. Prove the converse of Proposition 26.

11. ***On a given straight line as chord, construct a segment of a circle containing an angle equal to a given angle.***

Circles that Touch Each Other

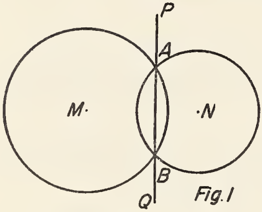


Fig. 1

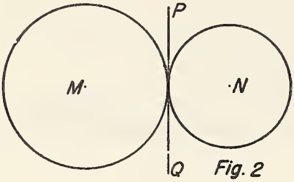


Fig. 2

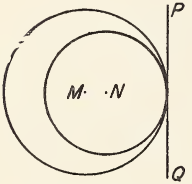


Fig. 3

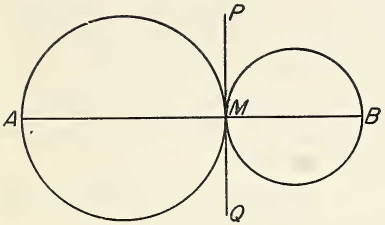
In the adjacent figure, the common chord AB of the two intersecting circles is produced to P and Q.

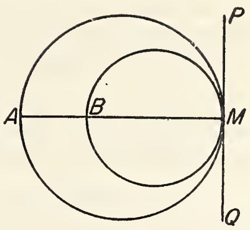
If the circle with centre M remains stationary, and the circle with centre N moves to the right, the points A and B eventually coincide and the secant PQ becomes a tangent to both circles (Fig. 2).

If the circle with centre M remains stationary, and the circle with centre N moves to the left, the figure takes the form shown in Fig. 3.

In Fig. 2, the circles are said to touch ***externally.***
In Fig. 3, they touch ***internally.***

DEFINITION: *If two circles meet at a point and have a common tangent at that point they are said to be* ***tangent to each other*** *at that point.* In Figures 2 and 3, the common tangent PQ is drawn at the point where the two circles touch.





In the foregoing figures AB is drawn perpendicular to the common tangent PQ at the point of contact M.

Since $AM \perp PQ$,

AM passes through the centre of the circle on which A lies.

Since $BM \perp PQ$,

BM passes through the centre of the circle on which B lies.

But AM and MB are in the same straight line.

Hence, a perpendicular drawn to the common tangent at the point of contact passes through the centres of the circles.

Hence, if two circles touch each other, the line of centres passes through the point of contact.

There are two corollaries which should be noted:

Corollary 1: ***If two circles touch externally, the distance between their centres is equal to the sum of their radii.***

Corollary 2: ***If two circles touch internally, the distance between their centres is equal to the difference of their radii.***

The converses of these corollaries are also true.

EXERCISES

1. P is a given point on a given circle. What is the locus of the centres of all circles which touch the given circle at P? A formal proof is not required.

2. Draw a circle of given radius which will touch a given circle at a given point. How many solutions are there?

3. Draw a circle which will touch a given circle at a given point and have its centre on a given straight line. Under what condition is it impossible to obtain a solution?

4. Draw a circle which will pass through a given point P and touch a given circle at a given point. Discuss the situation that arises when P lies on the given circle.

5. Given the centre, draw a circle which will touch a given circle. How many solutions are there?

6. Draw three circles with radii 2.5 cm., 2.9 cm., and 3.8 cm., each of which touches the others externally.

7. Draw two circles of radii 2.7 cm. and 2.4 cm. to touch each other externally and to touch internally a circle of radius 6.1 cm.

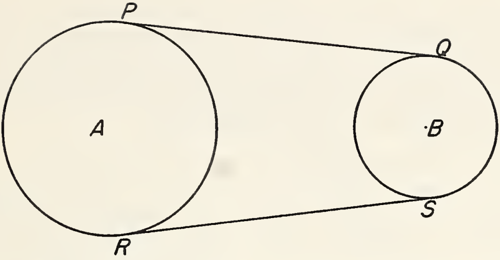
8. If two circles touch, any straight line through the point of contact forms segments which contain two pairs of equal angles.

9. Two circles with centres M and N touch, externally or internally, at P. A is any point on the circle with centre M. AP is joined and produced to meet the other circle at B. Show that $MA \parallel NB$.

10. Two equal circles with centres M and N touch externally at P. A is any point on the circle with centre M. AP when produced meets the other circle at B. (*a*) Show that $ANBM$ is a $\parallel$ gm. (*b*) Under what condition would $ANBM$ be a rectangle?

11. A circle of given radius moves so that it always touches a given circle. Find the locus of the centre. A formal proof is not required.

DEFINITION: *Circles which have the same centre are said to be* ***concentric circles.***



12. PQ and RS are the direct common tangents to the two given circles. Try to discover how to draw PQ and RS.

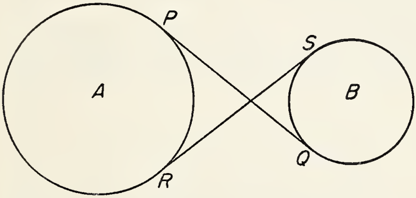
* * *

If you are completely baffled, the following hints may be helpful:

Join AP and BQ. What relationship exists between these two lines? * * *

What kind of figure would be formed if you draw $BT \parallel QP$ to meet AP at T?

Are the points A and B given? Is the line AB fixed in length and position? What is the size of $\angle ATB$? On what curved line must T lie? What is the length of AT? * * * Can you find T? Now you should be able to draw PQ. (What is the relation of BT to the *small* circle with centre A?)



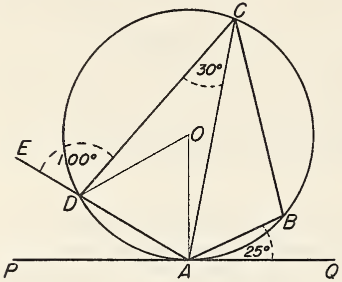
13. PQ and RS are the transverse common tangents. You are challenged to discover how to draw them. (This time, we shall not spoil your fun.)

REVIEW EXERCISES

A

1. With the aid of a figure, give the meaning of each of the following terms: (*a*) arc of a circle; (*b*) major and minor arcs of a circle; (*c*) chord of a circle; (*d*) segment of a circle; (*e*) angle in the segment of a circle; (*f*) angle at the centre of a circle; (*g*) reflex

angle at the centre of a circle; (*h*) circumcircle of a triangle; (*i*) inscribed circle of a triangle; (*j*) escribed circle of a triangle; (*k*) secant; (*l*) tangent; (*m*) circles touching externally; (*n*) circles touching internally; (*o*) concyclic points; (*p*) cyclic quadrilaterals; (*q*) concurrent lines; (*r*) concentric circles.



2. PAQ is a tangent at A to the circle with centre O. Find the number of degrees in each of the following angles: (*a*) $\angle ACB$; (*b*) $\angle DOA$; (*c*) $\angle ABC$; (*d*) $\angle PAD$; (*e*) $\angle OAC$; (*f*) $\angle OAD$; (*g*) $\angle ODA$; (*h*) $\angle OAP$.

3. With centres P and Q and radius PQ, two circles are drawn intersecting at M and N. Prove that (*a*) MPQ is an equilateral triangle; (*b*) $NQMP$ is a rhombus.

4. $ABCD$ is a cyclic quadrilateral in which $AB = CD$. Show that AD is parallel to BC.

5. Draw a circle of radius 2.5 cm. Draw PA, a tangent at A. Draw a chord AB making $\angle PAB$ equal to 45°. Draw a diameter BC. Measure (*a*) AB; (*b*) AC; (*c*) $\angle C$; (*d*) $\angle B$. Explain your result.

6. If two chords of a circle bisect each other, what must be true of them?

7. If two chords of a circle bisect each other at right angles, what figure is formed when the ends of the chords are joined? Give a proof.

8. A and B are the centres of two equal circles. $XY \parallel AB$ cuts the circle with centre A at M and N, and cuts the second circle at P and Q. M, N, P and Q occur in this order on the line. Prove that (*a*) $MN = PQ$; (*b*) $ABQN$ is a $\parallel$ gm; (*c*) $ABPM$ is a $\parallel$ gm; (*d*) $\parallel$ gm $ABQN = \parallel$ gm $ABPM$.

B

9. If the sides of a quadrilateral $ABCD$ are tangents to a circle, prove that $AB + CD = AD + BC$.

10. If the sides of a quadrilateral $ABCD$ are tangents to a circle whose centre is O, prove that $\angle AOB + \angle COD = 180°$.

11. If a triangle is inscribed in a circle, prove that the sum of the angles in the three segments outside the triangle is 360°.

12. $ABCD$ is a cyclic quadrilateral. A circle is drawn through A and B cutting DA and CB produced at M and N respectively. Show that $MN \parallel DC$.

13. Two circles touch externally at P. A is a point on one of the circles and AP produced meets the other circle at B. Prove that the tangents at A and B are parallel.

14. PA and PB are tangents to a circle, A and B being the points of contact. If the angle in the major arc of which AB is a chord is $(180 - 2\alpha)°$, find in terms of α the size of $\angle APB$.

15. (*a*) Construct a parallelogram such that its sides are tangents to a given circle.

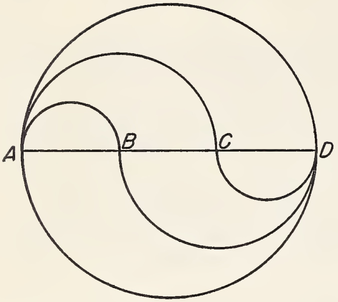
(*b*) Show that the parallelogram must be a rhombus.

16. Find the locus of the midpoints of chords drawn from a fixed point on a given circle. A formal proof is not required.

17. Given the hypotenuse and one of the other sides, construct a right triangle.

C

18. Divide a circle into two segments such that the angle in one is a third of the angle in the other.



19. In the adjacent figure, AD is a diameter of the circle and $AB = BC = CD$. Semicircles are described on AB, AC, BD and CD. Show that the lengths of the four curved lines joining A and D are equal.

20. If a chord of a circle is perpendicular to another chord, prove that the sum of each pair of opposite arcs is equal to a semicircle.

21. AB is a diameter of a circle with centre O. At A and B tangents are drawn to the circle. CD is also a tangent and intersects the other two tangents at C and D. Prove that $\angle DOC = 90°$.

22. Draw a circle which will pass through a given point and touch a given circle at a given point. Under what condition is it impossible to obtain a solution?

23. Two circles which touch externally at A have a common tangent which touches one of them at P and the other at Q. Show that $\angle PAQ = 90°$.

24. Construct a $\triangle ABC$ in which $BC = 7.5$ cm., $\angle B = 30°$, and radius of circumcircle $= 4.9$ cm. Measure AB and $\angle A$.

25. Draw a circle which touches a given circle, has its centre on a given straight line and passes through a given point in the given line.

26. (*a*) Using compasses and straight edge, construct on a line 2 inches in length a segment of a circle which contains an angle of 30°.

(*b*) AB is a line of given length. What is the locus of a point P which moves so that $\angle APB = 30°$?

QA 484 B78
BOWERS HENRY
LOCUS AND THE CIRCLE SECTION
B MATHEMATICS 20
NL 40815406 CURR HIST
000033706193

(*c*) Construct $\triangle ABC$ in which $BC = 3$ inches, and $\angle A = 45°$, and of which the area is 3 sq. inches.

27. Construct $\triangle ABC$ in which $BC = 3$ in., $AB = 3\frac{1}{8}$ in., and $\angle A = 47°$.

28. A point P moves so that the angle subtended at it by a given straight line AB is 60°. Draw the locus of the centre of the inscribed circle of $\triangle PAB$.

29. AB is a line of given length. A point P moves so that $\angle APB = 40°$. $AD \perp PB$ and $BE \perp PA$. AD and BE intersect at O. Find the locus of O. A formal proof is not required.

30. Draw a circle to touch a given circle and to touch a given straight line at a given point.